The Formal Theory of Tangentads

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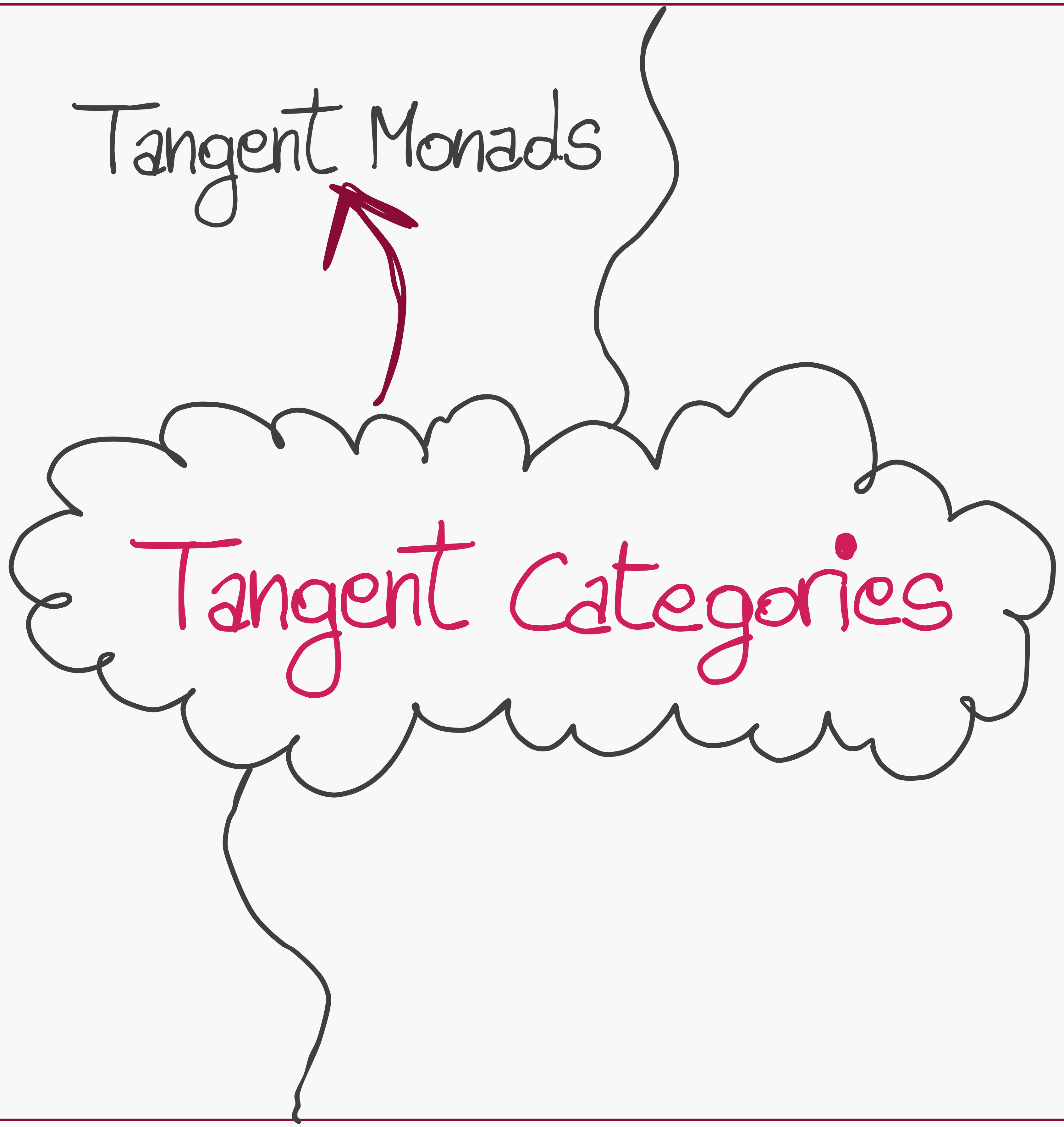
Macquarie · University

Kyoto · University 04 · 26

MANY FLAVOURS
MANY CONSTRUCTIONS

MOTIVAITION

MANY FLAVOURS



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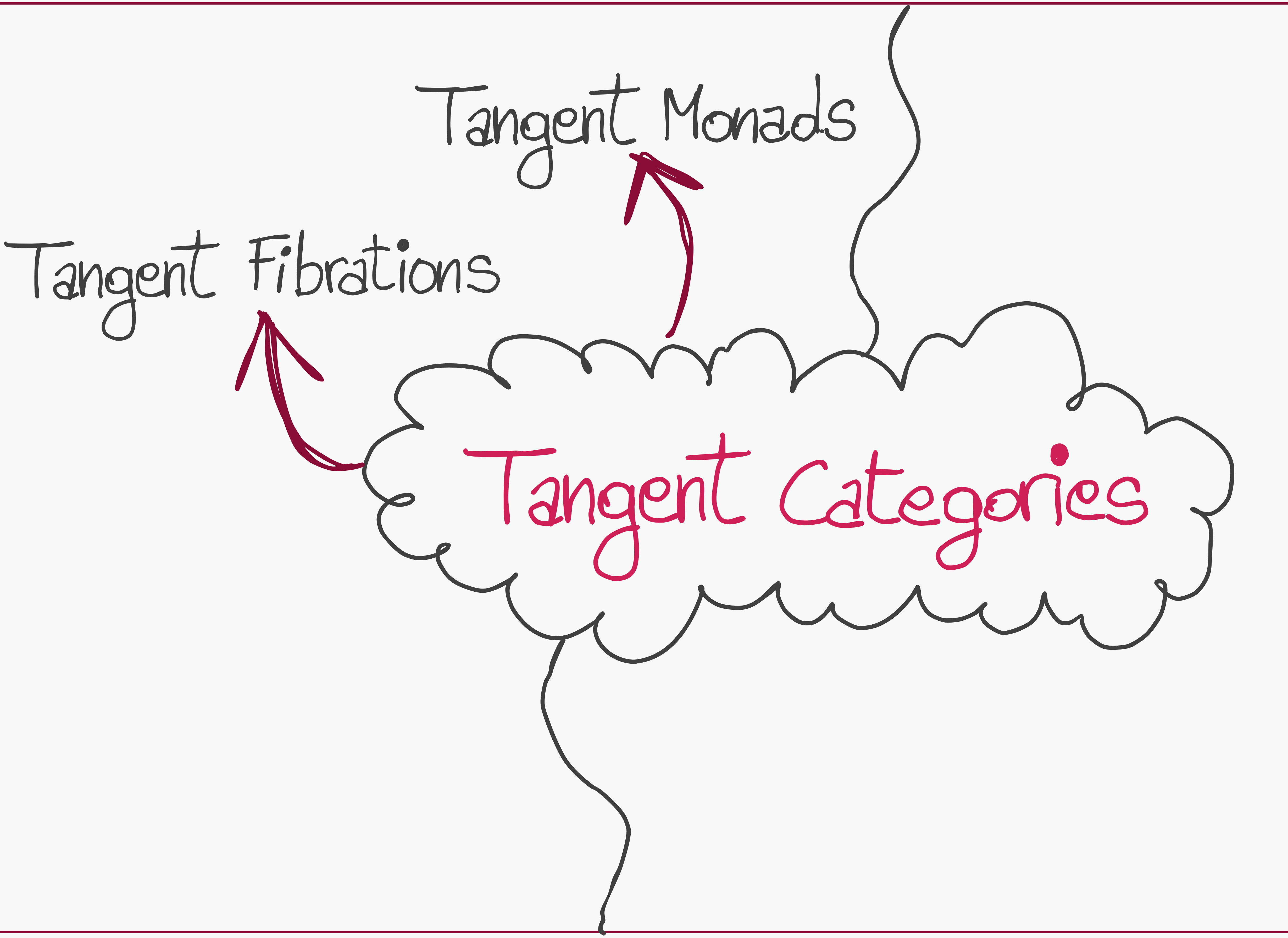

MOTIVAITION

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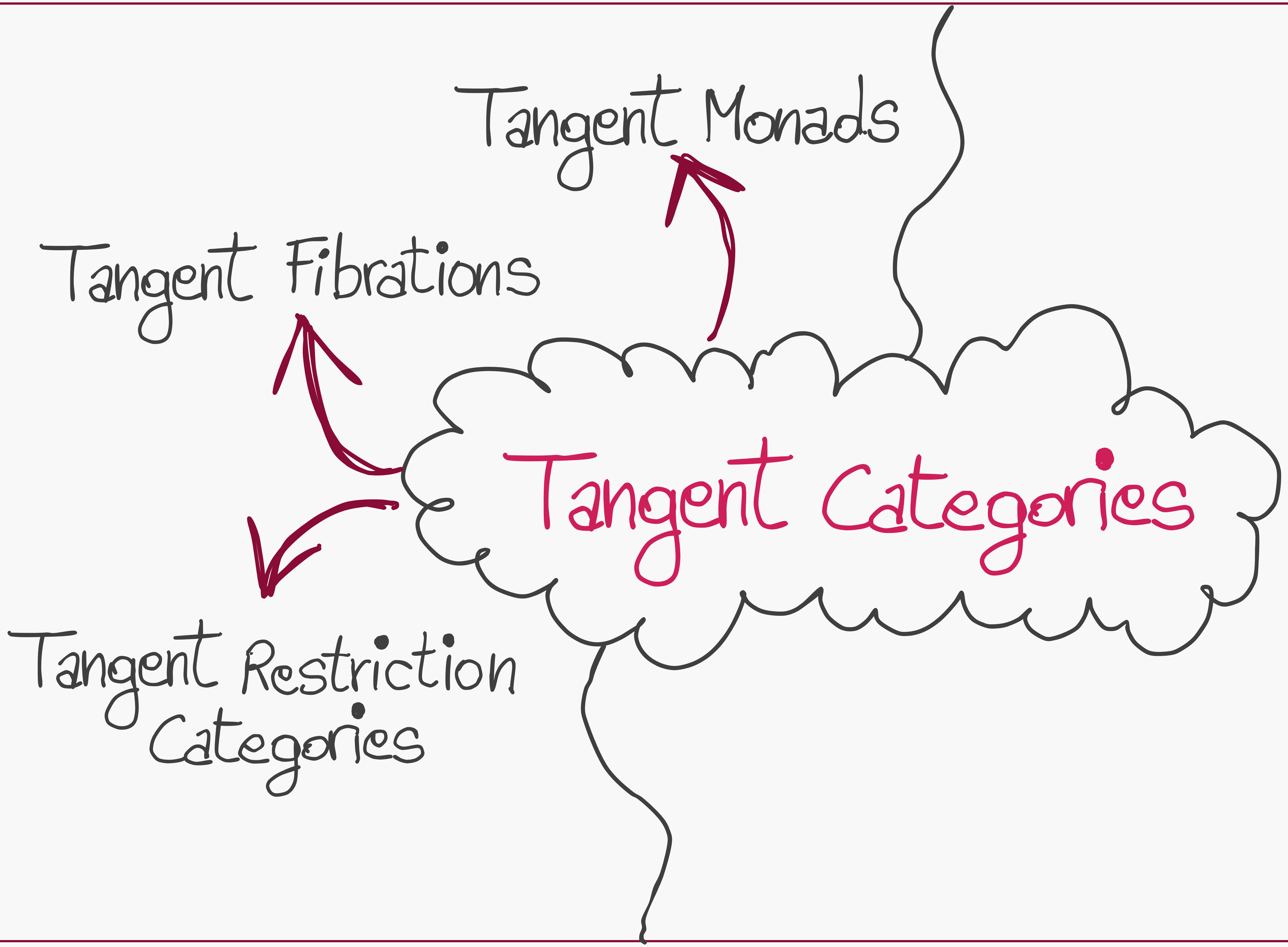

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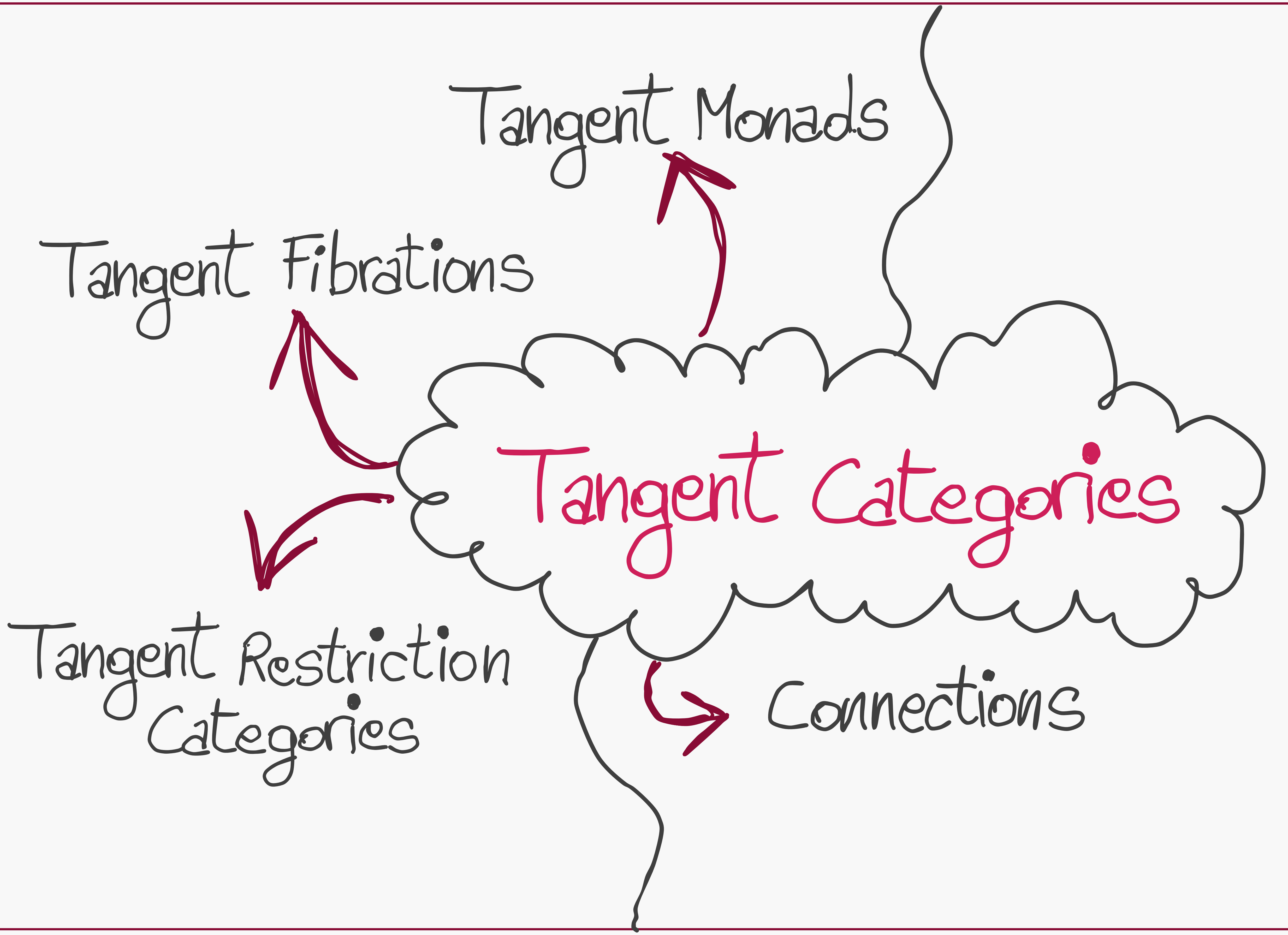
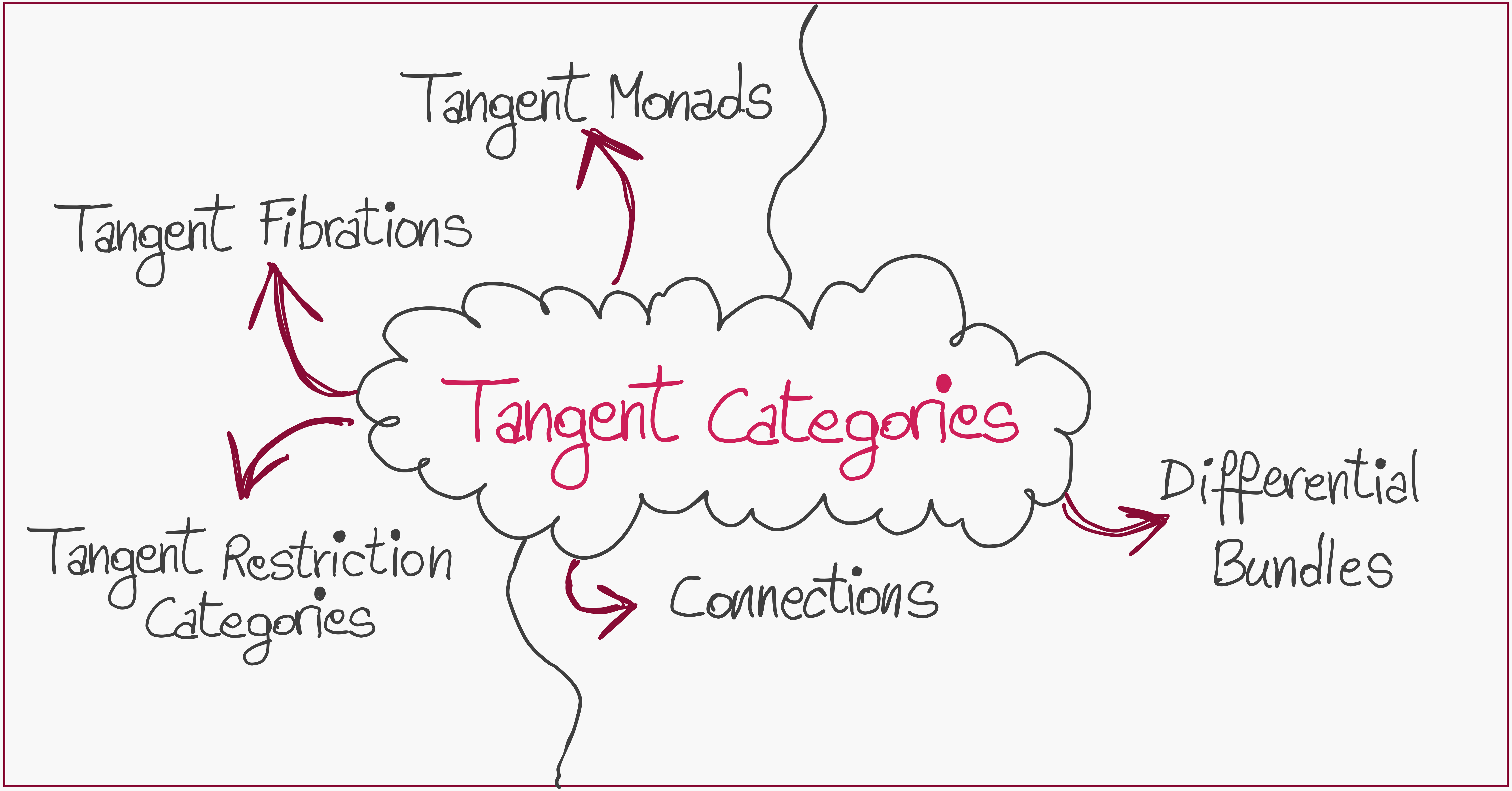
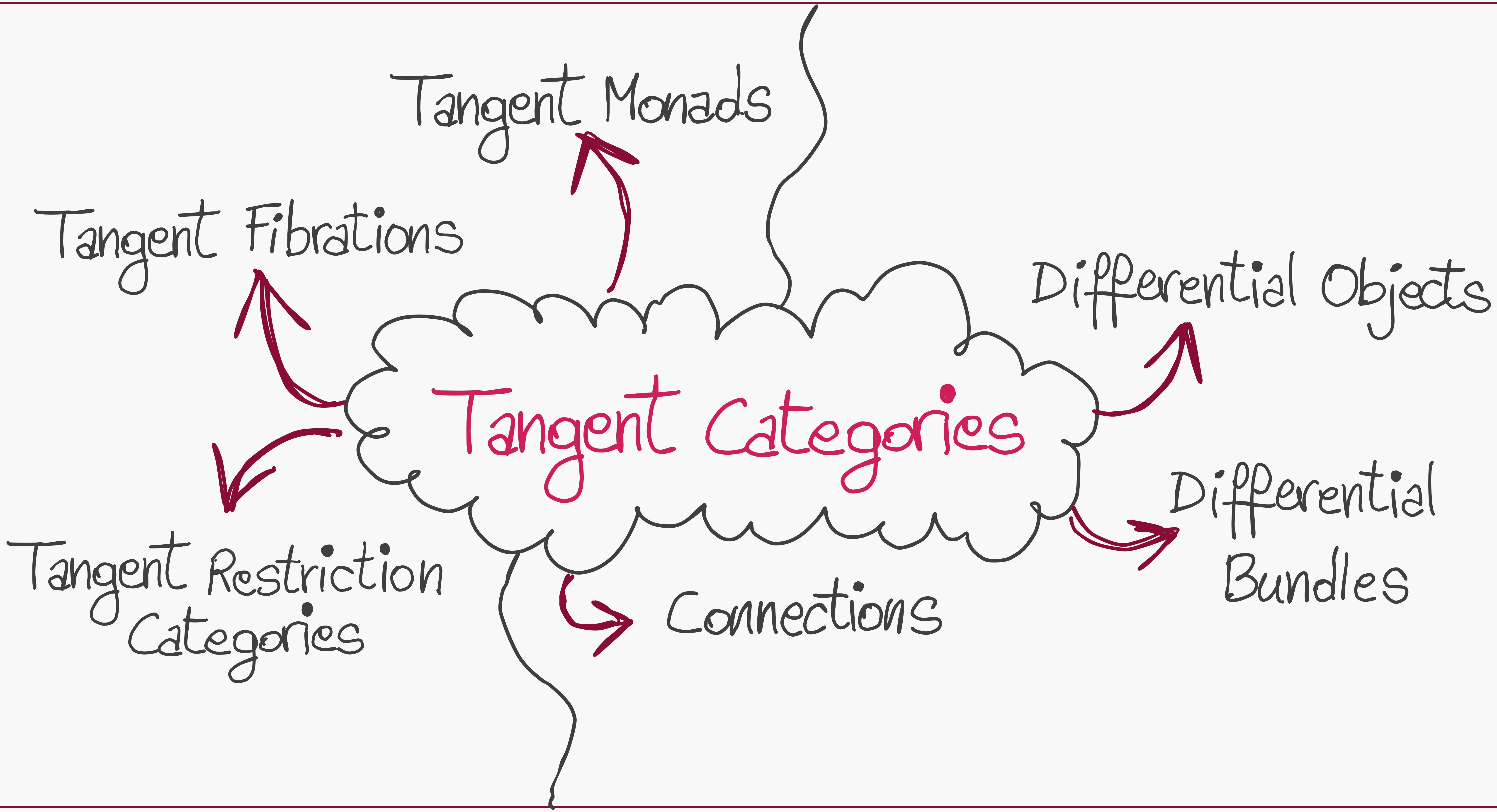


MANY CONSTRUCTIONS

MOTIVAITION



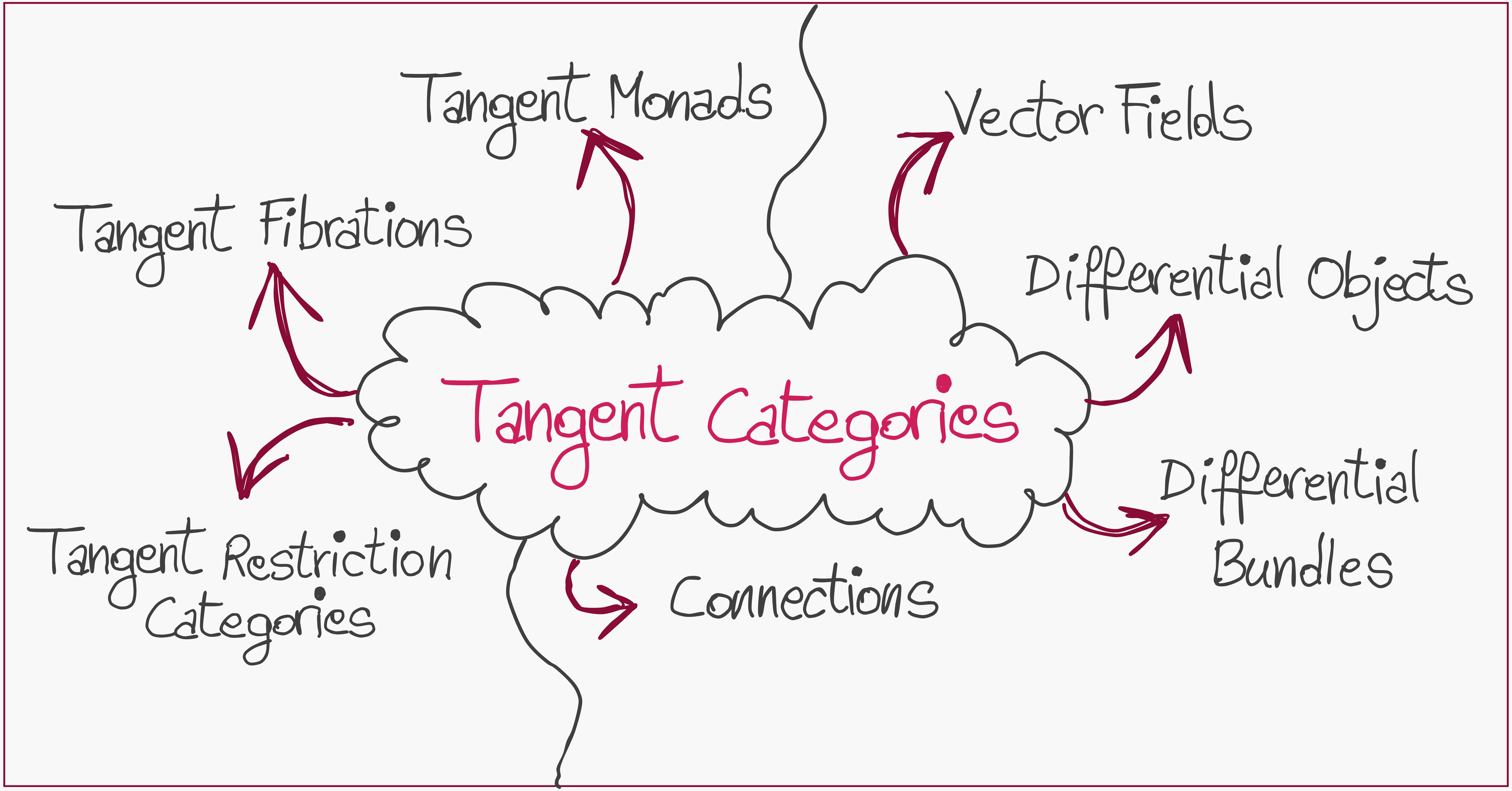
MANY CONSTRUCTIONS



Rosický 1984

MANY CONSTRUCTIONS

MOTIVAITION



MANY FLAVOURS · MANY CONSTRUCTIONS

MOTIVAITION

CAN WE FIND A UNIFYING THEORY?

THE PLAN FOR TODAY

TANGENT CATEGORIES

THE PLAN FOR TODAY

TANGENT CATEGORIES

TANGENTADS & EXAMPLES

THE PLAN FOR TODAY

TANGENT CATEGORIES

TANGENTADS & EXAMPLES

FORMAL VECTOR FIELDS

THE PLAN FOR TODAY

- **T**ANGENT CATEGORIES
- **T**ANGENTADS & EXAMPLES
- **F**ORMAL VECTOR FIELDS
- **L**IE ALGEBRA OF FORMAL VECTOR FIELDS

CHAPTER · ONE

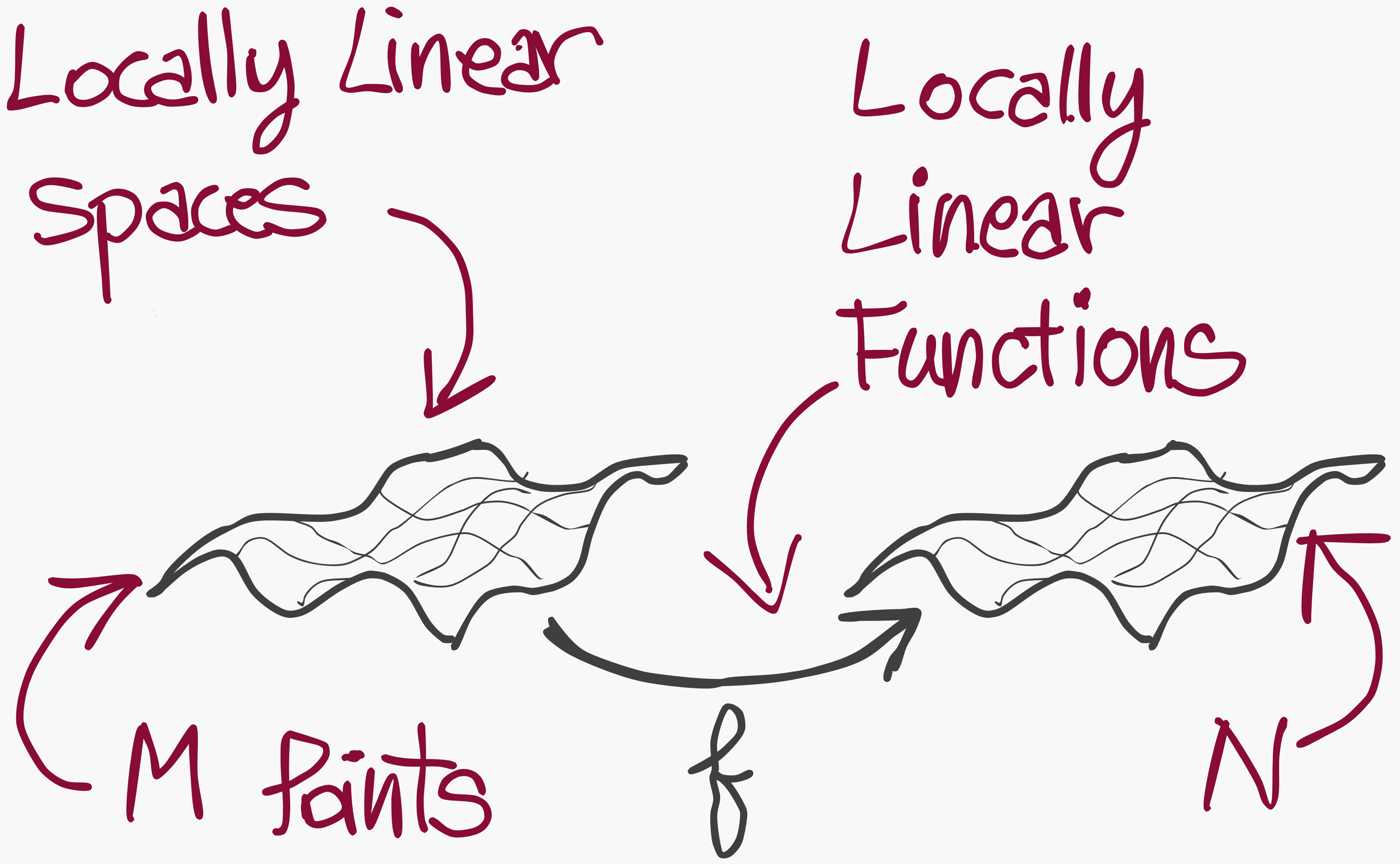
TANGENT CATEGORIES

A CATEGORICAL CONTEXT FOR DIFFERENTIAL GEOMETRY

TANGENT CATEGORY

DEFINITION

$\mathbb{X}$ Category



DEFINITION

TANGENT CATEGORY

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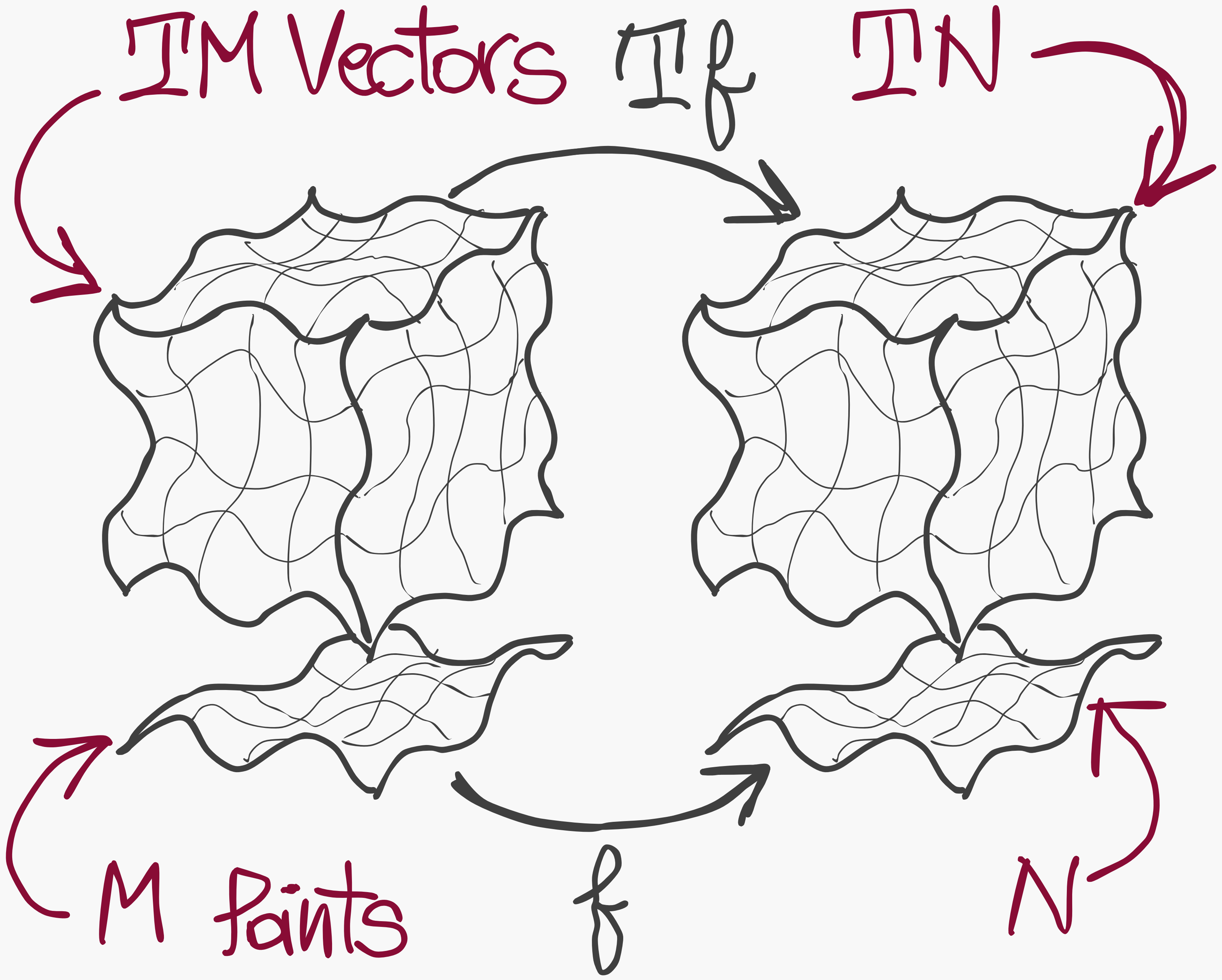
$\mathbb{X}$ Category

$\mathrm{T}: \mathbb{X} \to \mathbb{X}$ Tangent Bundle Functor



TANGENT CATEGORY

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DEFINITION

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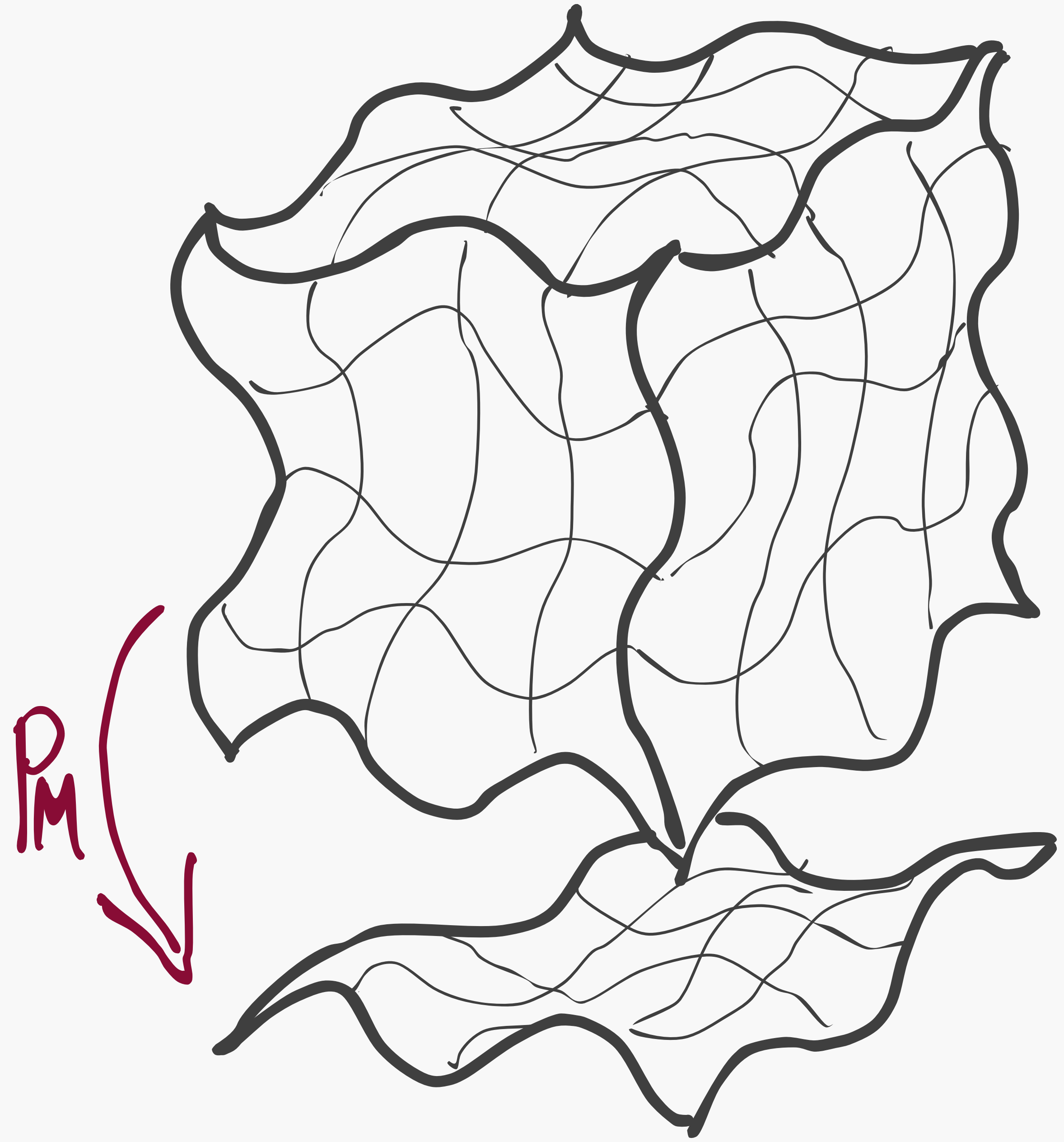
$T : \mathbb{X} \to \mathbb{X}$ Tangent Bundle Functor

$p_M : TM \to M$ Projection



TANGENT CATEGORY

Rosický 1984 Cockett Cruttwell 2014

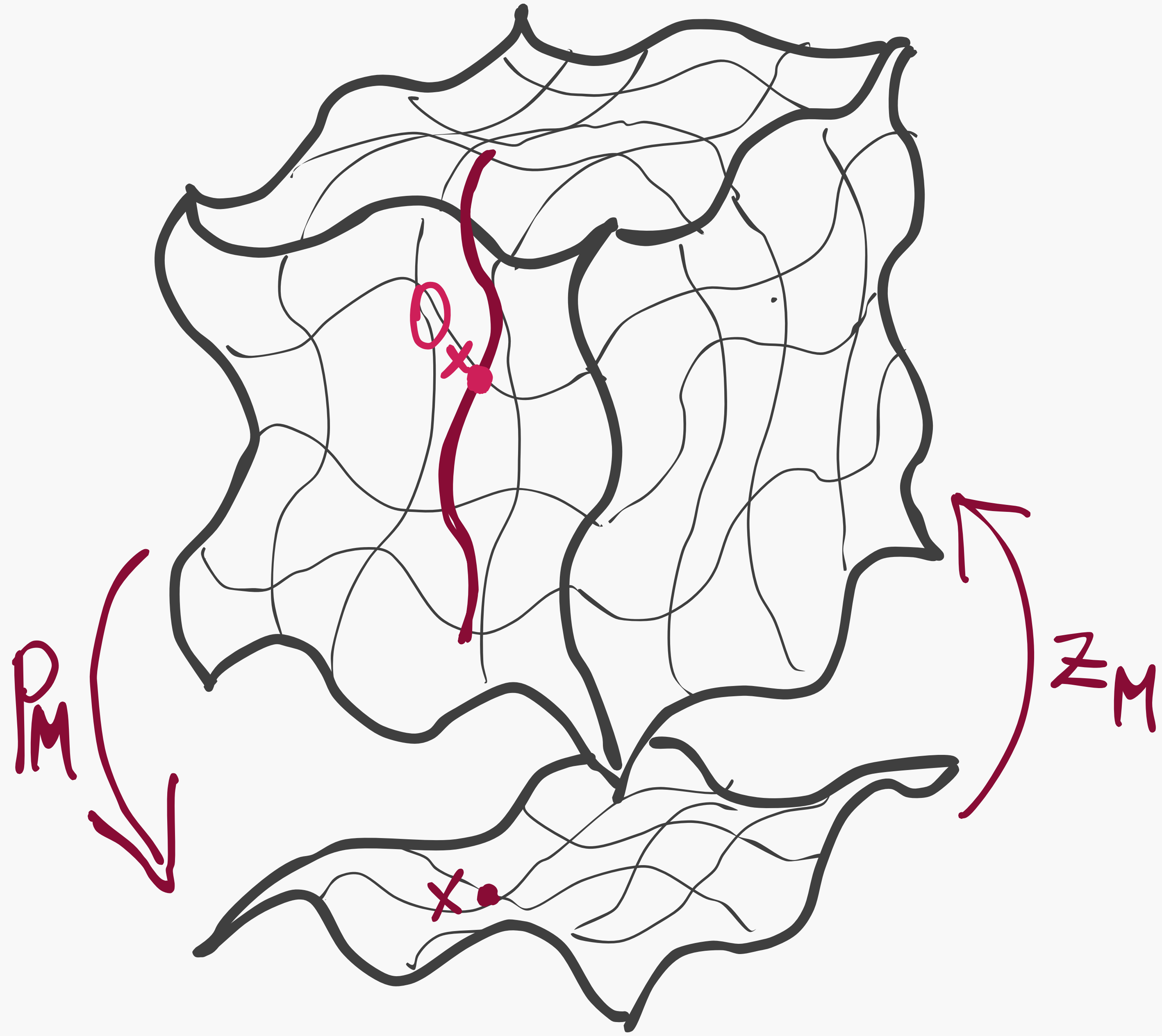
DEFINITION

$\mathbb{X}$ Category

$\mathbb{T}: \mathbb{X} \to \mathbb{X}$ Tangent Bundle Functor

$p_M: \mathbb{T}M \to M$ Projection

$z_M: M \to \mathbb{T}M$ Zero



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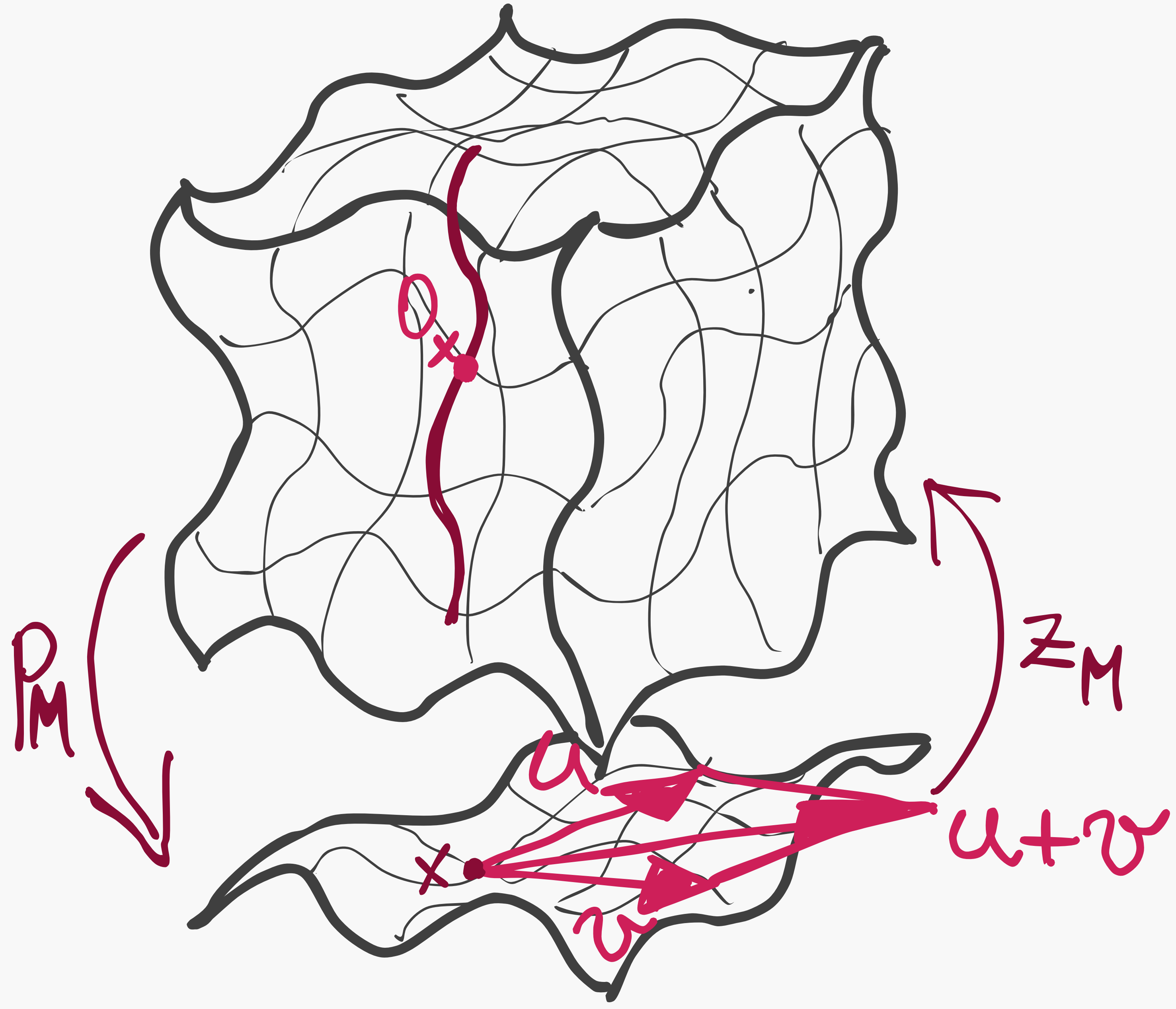

TANGENT CATEGORY

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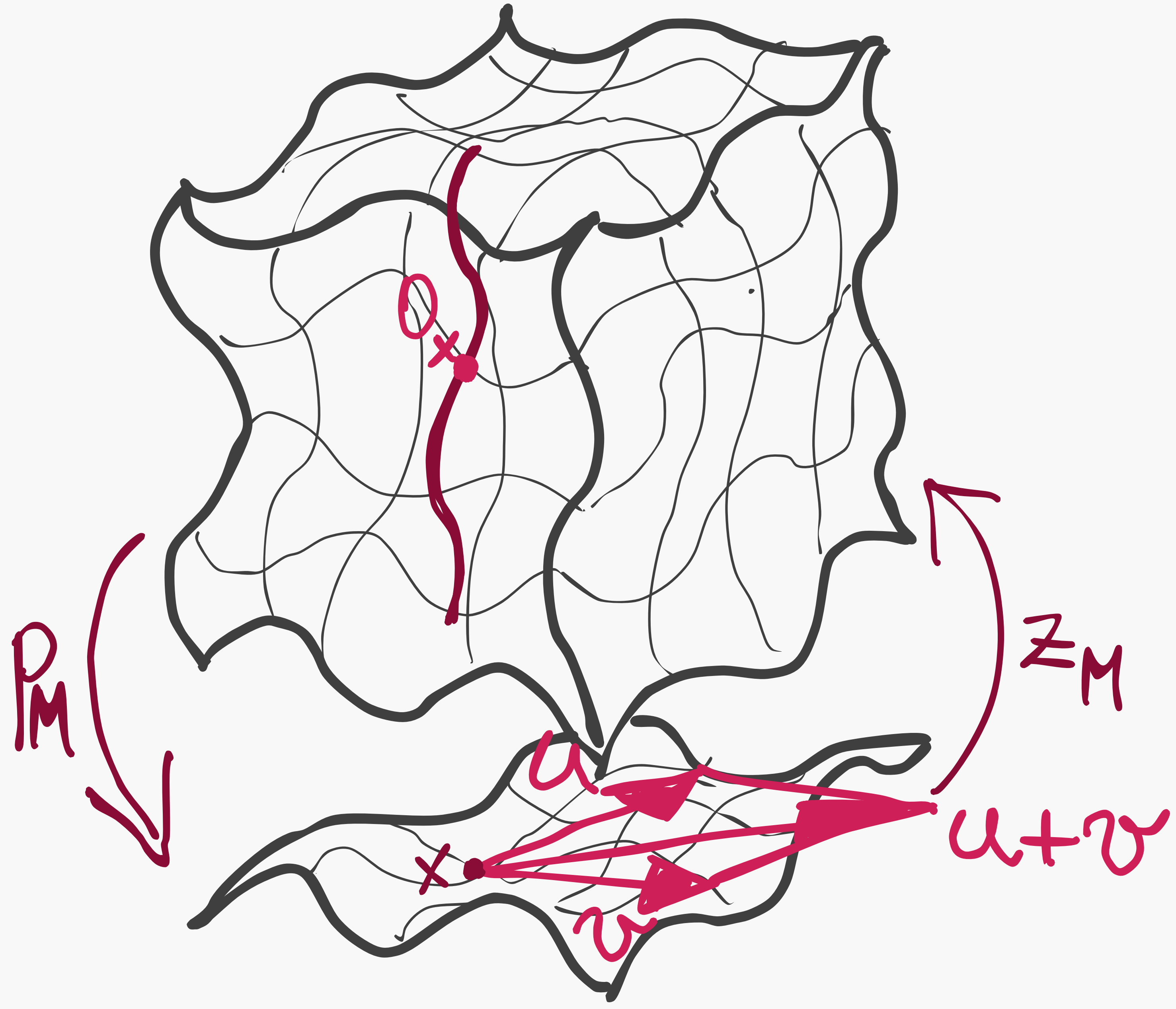
$s_M : T_2M \to TM$ Sum

$$\begin{tikzcd} T_2M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & TM \arrow[d, "p_M"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd}$$



TANGENT CATEGORY

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$e_M : TM \to TTM$ Vertical Lift

Encodes Local Linearity

TANGENT CATEGORY

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DEFINITION

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$c_M : TTM \to TTM$ Canonical Flip

Encodes Local Linearity

Symmetry of Directional Derivatives

$\partial_u \partial_v f = \partial_v \partial_u f$

TANGENT CATEGORY

Rosický 19**84** **C**ockett Cruttwell 20**14**

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Encodes Local Linearity

Symmetry of Directional Derivatives

$\partial_u \partial_v f = \partial_v \partial_u f$

+ Axioms

CHAPTER · TWO

TANGENTADS & EXAMPLES

TANGENT CATEGORIES ARE TO MONADS
AS TANGENTADS ARE TO FORMAL MONADS

TANGENT CATEGORY

DEFINITION

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$p_M : \mathrm{T}M \to M$ Projection

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$s_M : \mathrm{T}_2 M \to \mathrm{T}M$ Sum

$\ell_M : \mathrm{T}M \to \mathrm{T}\mathrm{T}M$ Vertical Lift

$c_M : \mathrm{T}\mathrm{T}M \to \mathrm{T}\mathrm{T}M$ Canonical Flip

TANGENT CATEGORY · TANGENTAD

DEFINITION

TANGENT CATEGORY	TANGENTAD	
$\mathbb{X}$ Category	$\mathbb{X}$ Object in a 2-category	
$\mathrm{T}:\mathbb{X}\to\mathbb{X}$ Tangent Bundle Functor	$\mathrm{T}:\mathbb{X}\to\mathbb{X}$ 1-morphism	
$p_M:\mathrm{T}M\to M$ Projection	$p:\mathrm{T}\Rightarrow \mathrm{id}_{\mathbb{X}}$	
$z_M:M\to\mathrm{T}M$ Zero	$z:\mathrm{id}_{\mathbb{X}}\Rightarrow\mathrm{T}$	2-morphisms
$s_M:\mathrm{T}_2M\to\mathrm{T}M$ Sum	$s:\mathrm{T}_2\Rightarrow\mathrm{T}$	
$e_M:\mathrm{T}M\to\mathrm{T}\mathrm{T}M$ Vertical Lift	$e:\mathrm{T}\Rightarrow\mathrm{T}\mathrm{T}$	
$c_M:\mathrm{T}\mathrm{T}M\to\mathrm{T}\mathrm{T}M$ Canonical Flip	$c:\mathrm{T}\mathrm{T}\Rightarrow\mathrm{T}\mathrm{T}$	

DEFINITION

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$\mathbb{X}$ Category	$\mathbb{X}$ Object in a 2-category
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$p_M: TM\to M$ Projection	$p: T\Rightarrow \mathrm{id}_{\mathbb{X}}$
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$s_M: T_2M\to TM$ Sum	$s: T_2\Rightarrow T$
$e_M: TM\to TTM$ Vertical Lift	$e: T\Rightarrow TT$
$c_M: TTM\to TTM$ Canonical Flip	$c: TT\Rightarrow TT$

2-morphisms

+ Axioms

THE LEUNG APPROACH

A tangent Category is
a strong monoidal functor

$$\mathcal{W}eil_1 \longrightarrow \mathcal{E}nd(\mathbb{X})$$

THE LEUNG APPROACH

ANOTHER POINT OF VIEW

A tangent category is a strong monoidal functor

$$\mathrm{Weil}_1 \longrightarrow \mathrm{End}(\mathbb{X})$$

endofunctors

$$W_n := \frac{\mathbb{N}[x_1, \ldots, x_n]}{(x_i x_j,\ i, j)}$$

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preserving some limits ptwise.

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A tangent ∞-category is a strong monoidal ∞-functor

$$\mathrm{Weil}_1^{\infty} \longrightarrow \mathcal{E}nd(\mathfrak{X})$$

preserving some limits ptwise.

THE LEUNG APPROACH

ANOTHER POINT OF VIEW

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A tangent ∞-category is a strong monoidal ∞-functor

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A tangentad is a strong monoidal functor

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preserving some limits ptwise.

EXAMPLES

Cat Tangent Categories	

EXAMPLES

Cat Tangent Categories

Mnd[$\mathbb{K}$] Tangent Monads

TANGENTADS

EXAMPLES

Cat Tangent Categories

Mnd[$\mathbb{K}$] Tangent Monads

Fib Tangent Fibrations

TANGENTADS

EXAMPLES

Cat	Tangent Categories
Mnd[$\mathbb{K}$]	Tangent Monads
Fib	Tangent Fibrations
sRCat	Tangent Split Restriction Categories

TANGENTADS

EXAMPLES

Cat	Tangent Categories
Mnd[$\mathbb{K}$]	Tangent Monads
Fib	Tangent Fibrations
sRCat	Tangent Split Restriction Categories
$\mathrm{Span}^{op}[\mathbb{X}]$	Relative cotangentoids (Infinitesimal Objects)

CHAPTER · THREE

FORMAL VECTOR FIELDS

FORMALIZING GEOMETRY

VECTOR FIELDS IN TANGENT CATS

DEFINITION

A vector field in a tangent category is a section of p:

$v : M \to TM$

$$\begin{tikzcd} & TM \arrow[dr, "p_M"] & \\ M \arrow[ur, "v"] \arrow[rr, equal] & & M \end{tikzcd}$$

VECTOR FIELDS IN TANGENT CATS

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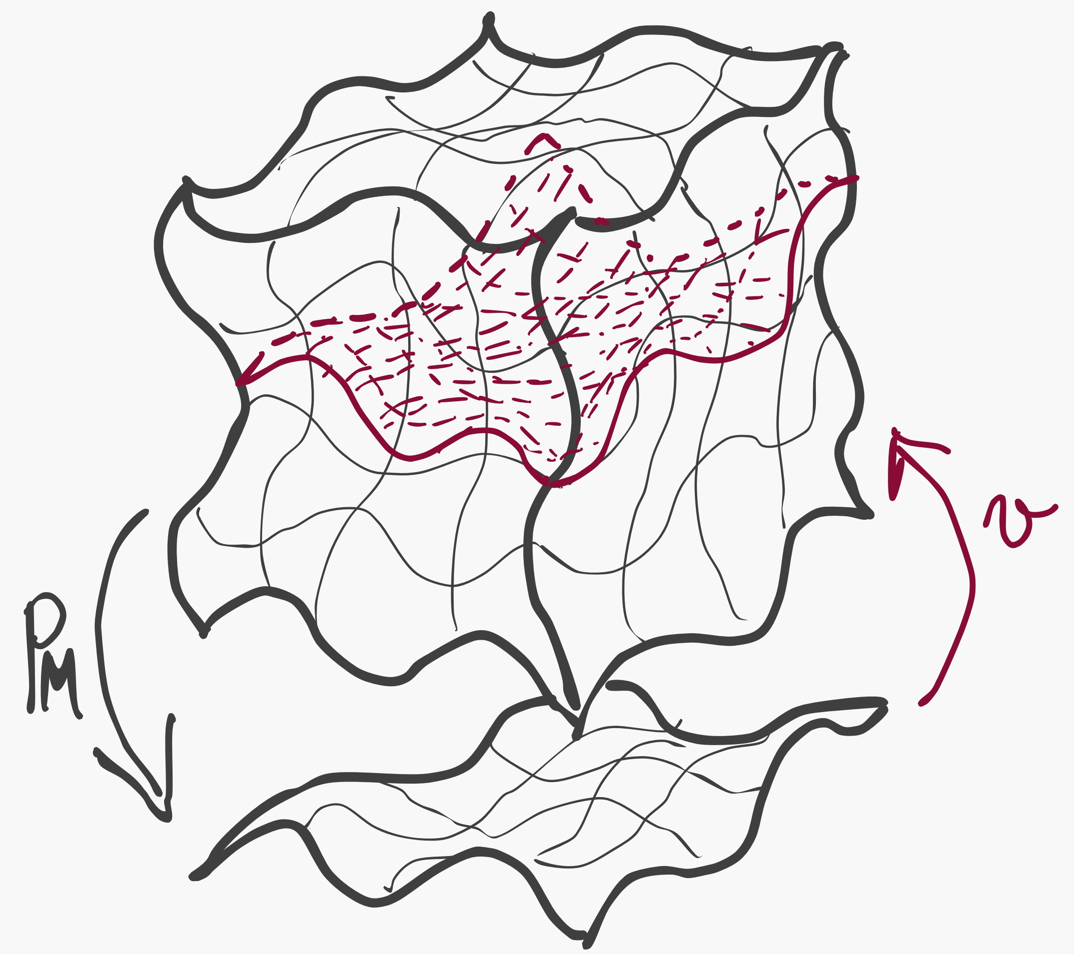

VECTOR FIELDS IN TANGENT CATS

THEOREM

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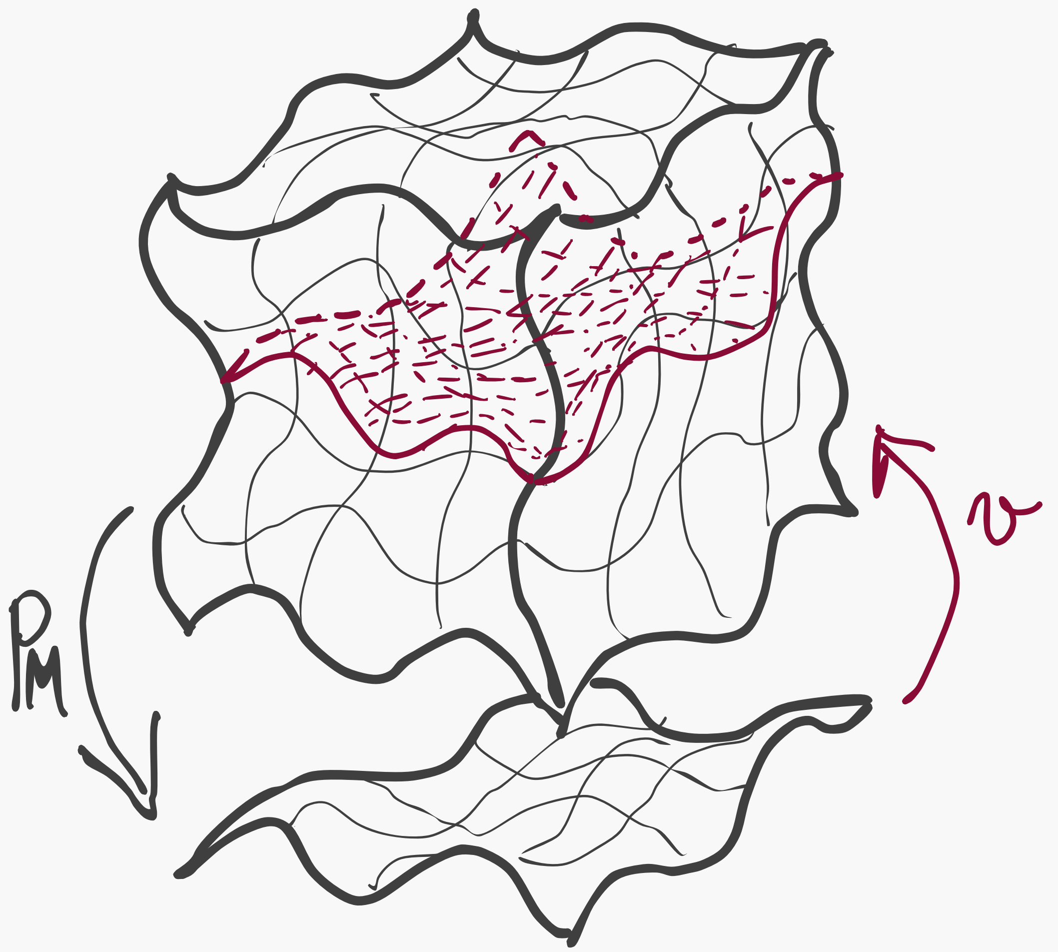

With negatives, vector fields over a fixed object form a Lie algebra.

$[\,,\,] : VF_M(\mathbb{X},\mathbb{T}) \times VF_M(\mathbb{X},\mathbb{T}) \to VF_M(\mathbb{X},\mathbb{T})$

VECTOR FIELDS IN TANGENT CATS

THEOREM

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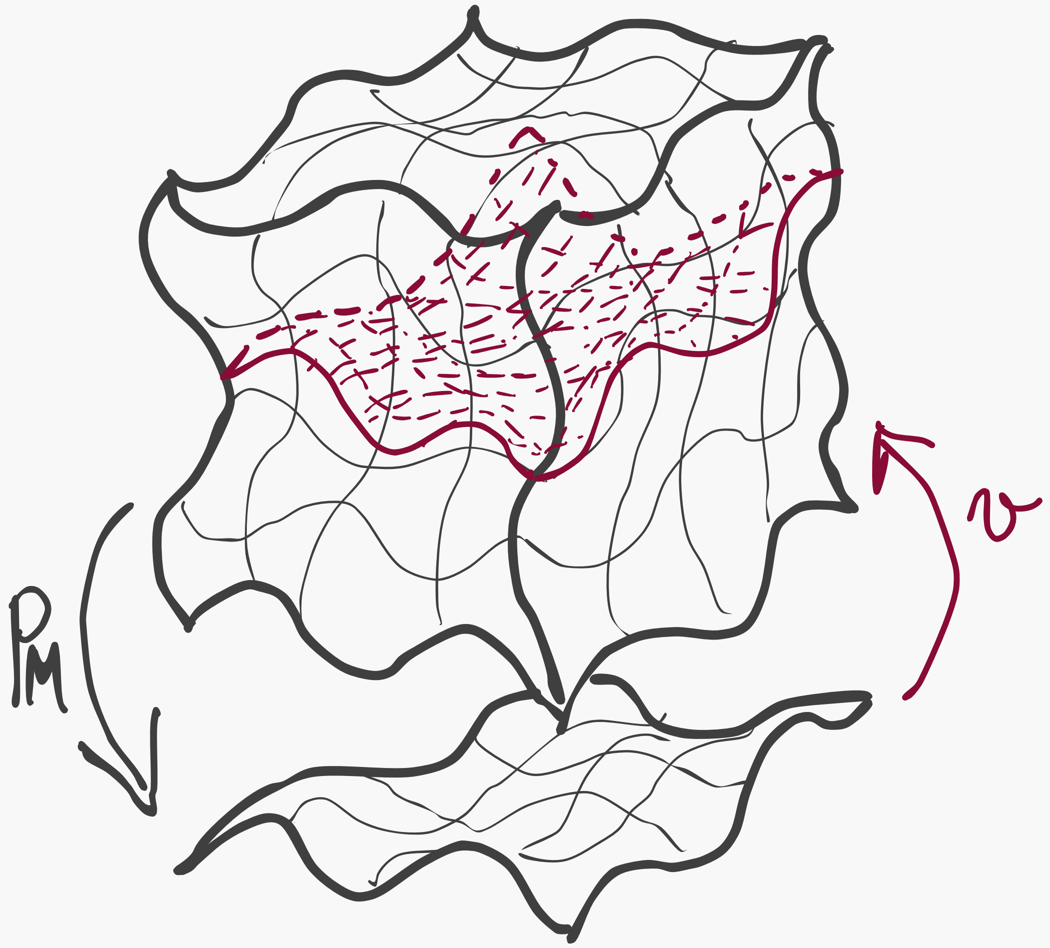

With negatives, vector fields over a fixed object form a Lie algebra.

$[,] : VF_M(\mathbb{X},\mathbb{T}) \times VF_M(\mathbb{X},\mathbb{T}) \to VF_M(\mathbb{X},\mathbb{T})$

Moreover, vector fields form a tangent category:

$VF(\mathbb{X},\mathbb{T})$

$\mathbb{T}^{VF}(M,v) = (TM, TM \xrightarrow{Tv} T^2M \xrightarrow{c_M} TM)$

TOWARDS FORMAL VECTOR FIELDS

CONSTRUCTION

$$VF(\mathfrak{X}, \mathbb{T}) \xrightarrow{\mathcal{U}} (\mathfrak{X}, \mathbb{T})$$

$$(M, v) \longmapsto M$$

TOWARDS FORMAL VECTOR FIELDS

CONSTRUCTION

$$\mathrm{VF}(\mathbb{X},\mathbb{T}) \xrightarrow{\mathcal{U}} (\mathbb{X},\mathbb{T})$$

$$(M,v) \longmapsto M$$

$$\frac{\hat{v}_{(M,v)} : \mathcal{U}(M,v) \to \mathbb{T}(\mathcal{U}(M,v))}{M \xrightarrow{v} \mathbb{T}M}$$

TOWARDS FORMAL VECTOR FIELDS

CONSTRUCTION

$$\mathrm{VF}(\mathbb{X},\mathbb{T}) \xrightarrow{U} (\mathbb{X},\mathbb{T})$$
$$(M,v) \longmapsto M$$

$$\frac{\hat{v}_{(M,v)} : U(M,v) \longrightarrow \mathbb{T}(U(M,v))}{M \xrightarrow{v} \mathbb{T}M}$$

□ $\hat{v}$ is natural:

$$\frac{f:(M,v) \to (N,u)}{\begin{array}{ccc} \mathbb{T}M & \xrightarrow{\mathbb{T}f} & \mathbb{T}N \\ v\uparrow & & \uparrow u \\ M & \xrightarrow{f} & N \end{array}}$$

TOWARDS FORMAL VECTOR FIELDS

CONSTRUCTION

$$\mathrm{VF}(\mathbb{X},\mathbb{T}) \xrightarrow{U} (\mathbb{X},\mathbb{T})$$

$$(M,v) \longmapsto M$$

$$\frac{\hat{v}_{(M,v)} : U(M,v) \longrightarrow \mathbb{T}(U(M,v))}{M \xrightarrow{v} \mathbb{T}M}$$

□ $\hat{v}$ is natural:

$$\frac{f:(M,v)\to(N,u)}{\begin{tikzcd} \mathbb{T}M \arrow[r, "\mathbb{T}f"] & \mathbb{T}N \\ M \arrow[u, "v"] \arrow[r, "f"] & N \arrow[u, "u"'] \end{tikzcd}}$$

□ $\hat{v}$ is a "vector field":

$$\begin{tikzcd} & \mathbb{T}U(M,v) \arrow[dr, "p"] & \\ U(M,v) \arrow[ur, "\hat{v}"] \arrow[rr, equal] & & U(M,v) \end{tikzcd}$$

THE HOM-TANGENT CATEGORIES

PROPOSITION

The Hom-category

$[\mathbb{X},\mathbb{T}|\mathbb{X}',\mathbb{T}']$

comes with a tangent structure

$\overline{\mathbb{T}}(F,\alpha):(\mathbb{X},\mathbb{T})\xrightarrow{(F,\alpha)}(\mathbb{X}',\mathbb{T}')\xrightarrow{(\mathrm{T}',c')}(\mathbb{X}',\mathbb{T}')$

THE HOM-TANGENT CATEGORIES

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$$\frac{(F,\alpha):(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}')}{\begin{array}{l}F:\mathbb{X}\to\mathbb{X}'\\ \alpha:F\circ\mathbb{T}\Rightarrow\mathbb{T}'\circ F\end{array}}$$

THE UNIVERSAL VECTOR FIELD

PROPOSITION

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$$\frac{(F,\alpha):(\mathbb{X},\mathbb{T})\longrightarrow(\mathbb{X}',\mathbb{T}')}{\begin{array}{l}F:\mathbb{X}\longrightarrow\mathbb{X}'\\ \alpha:F\circ\mathrm{T}\Rightarrow\mathrm{T}'\circ F\end{array}}$$

$\mathcal{U}:\mathrm{VF}(\mathbb{X},\mathbb{T})\longrightarrow(\mathbb{X},\mathbb{T})$

with $\hat{v}:\mathcal{U}\Rightarrow\mathbb{T}\mathcal{U}$

is a vector field in

$[\mathrm{VF}(\mathbb{X},\mathbb{T})|\mathbb{X},\mathbb{T}]$.

THE UNIVERSAL VECTOR FIELD

PROPOSITION

The Hom-category

$[\mathbb{X},\mathbb{T}|\mathbb{X}',\mathbb{T}']$

comes with a tangent structure

$$\mathbb{T}(F,\alpha):(\mathbb{X},\mathbb{T}) \xrightarrow{(F,\alpha)} (\mathbb{X}',\mathbb{T}') \xrightarrow{(\mathbb{T}',c)} (\mathbb{X}',\mathbb{T}')$$

$$\frac{(F,\alpha):(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X}',\mathbb{T}')}{\begin{array}{l} F:\mathbb{X} \longrightarrow \mathbb{X}' \\ \alpha: F\circ\mathbb{T} \Rightarrow \mathbb{T}'\circ F \end{array}}$$

$u: VF(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X},\mathbb{T})$

with $\hat{v}: u \Rightarrow \mathbb{T}u$

is a vector field in

$[VF(\mathbb{X},\mathbb{T})|\mathbb{X},\mathbb{T}]$.

This induces:

$$[\mathbb{X}',\mathbb{T}'|VF(\mathbb{X},\mathbb{T})] \to VF[\mathbb{X}',\mathbb{T}'|\mathbb{X},\mathbb{T}]$$

THE UNIVERSAL VECTOR FIELD

PROPOSITION

The Hom-category $[\mathbb{X},\mathbb{T}|\mathbb{X}',\mathbb{T}']$ comes with a tangent structure

$$\underbrace{\overline{\mathbb{T}}(F,\alpha)}:(\mathbb{X},\mathbb{T})\xrightarrow{(F,\alpha)}(\mathbb{X}',\mathbb{T}')\xrightarrow{(\mathrm{T}',c')}(\mathbb{X}',\mathbb{T}')$$

$$\frac{(F,\alpha):(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}')}{\begin{array}{l}F:\mathbb{X}\to\mathbb{X}'\\ \alpha:F\circ\mathrm{T}\Rightarrow\mathrm{T}'\circ F\end{array}}$$

$\mathcal{U}:\mathrm{VF}(\mathbb{X},\mathbb{T})\to(\mathbb{X},\mathbb{T})$ with $\hat{v}:\mathcal{U}\Rightarrow\overline{\mathbb{T}}\mathcal{U}$ is a vector field in $[\mathrm{VF}(\mathbb{X},\mathbb{T})|\mathbb{X},\mathbb{T}]$.

This induces:

$$[\mathbb{X}',\mathbb{T}'|\mathrm{VF}(\mathbb{X},\mathbb{T})]\to\mathrm{VF}[\mathbb{X}',\mathbb{T}'|\mathbb{X},\mathbb{T}]$$

$$(F,\alpha)\mapsto\begin{tikzcd}(\mathbb{X}',\mathbb{T}')\arrow[r,"{(F,\alpha)}"] & \mathrm{VF}(\mathbb{X},\mathbb{T})\arrow[r,bend left,"\overline{\mathbb{T}}\mathcal{U}"]\arrow[r,bend right,"\mathcal{U}"']\arrow[r,phantom,"\Uparrow\hat{v}"] & (\mathbb{X},\mathbb{T})\end{tikzcd}$$

THE UNIVERSAL VECTOR FIELD

THEOREM

The functor

$$[\mathbb{X}',\mathbb{T}' \,|\, VF(\mathbb{X},\mathbb{T})] \to VF[\mathbb{X}',\mathbb{T}' \,|\, \mathbb{X},\mathbb{T}]$$

$$(F,\alpha) \mapsto \begin{tikzcd} (\mathbb{X}',\mathbb{T}') \arrow[r, "(F,\alpha)"] & VF(\mathbb{X},\mathbb{T}) \arrow[r, bend left, "\mathbb{T}u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Uparrow \hat{v}"] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

induced by $(u,\hat{v})$ is an isomorphism.

THE UNIVERSAL VECTOR FIELD

DEFINITION

The functor

$$[\mathbb{X}',\mathbb{T}' \mid \mathsf{VF}(\mathbb{X},\mathbb{T})] \to \mathsf{VF}[\mathbb{X}',\mathbb{T}' \mid \mathbb{X},\mathbb{T}]$$

$$(F,\alpha) \mapsto \begin{tikzcd} (\mathbb{X}',\mathbb{T}') \arrow[r, "{(F,\alpha)}"] & \mathsf{VF}(\mathbb{X},\mathbb{T}) \arrow[r, bend left, "\mathbb{T}u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Uparrow \hat{v}"] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

induced by $(u,\hat{v})$ is an isomorphism.

A tangentad $(\mathbb{X},\mathbb{T})$ admits the construction of vector fields provided the existence of a tangentad $\mathsf{VF}(\mathbb{X},\mathbb{T})$ equipped with a vector field $(u,\hat{v}) \in [\mathsf{VF}(\mathbb{X},\mathbb{T}) \mid \mathbb{X},\mathbb{T}]$ s.t.

$$[\mathbb{X}',\mathbb{T}' \mid \mathsf{VF}(\mathbb{X},\mathbb{T})] \to \mathsf{VF}[\mathbb{X}',\mathbb{T}' \mid \mathbb{X},\mathbb{T}]$$

is an isomorphism.

CHAPTER · FOUR

LIE ALGEBRA OF FORMAL VECTOR FIELDS

THE ZERO AND THE NEGATION

CONSTRUCTION

The zero:

$z : (F,\alpha) \Rrightarrow \mathbb{I}(F,\alpha)$ is a vect. field:

$z \in VF[\mathbb{X}',\mathbb{T}' | \mathbb{X},\mathbb{T}]$

THE ZERO AND THE NEGATION

CONSTRUCTION

The zero:

$Z : (F,\alpha) \Rrightarrow \mathbb{T}(F,\alpha)$ is a vect. field:

$Z \in VF[\mathbb{X}',\mathbb{T}' | \mathbb{X},\mathbb{T}]$

Take $(F,\alpha) = id_{(\mathbb{X},\mathbb{T})}$:

$Z \in VF[\mathbb{X},\mathbb{T} | \mathbb{X},\mathbb{T}] \cong [\mathbb{X},\mathbb{T} | VF(\mathbb{X},\mathbb{T})]$

THE ZERO AND THE NEGATION

CONSTRUCTION

The zero:

$z : (F,\alpha) \Rightarrow \mathbb{T}(F,\alpha)$ is a vect. field:

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Take $(F,\alpha) = id_{(\mathbb{X},\mathbb{T})}$:

$z \in VF[\mathbb{X},\mathbb{T} \mid \mathbb{X},\mathbb{T}] \cong [\mathbb{X},\mathbb{T} \mid VF(\mathbb{X},\mathbb{T})]$

$\Rightarrow 0 : (\mathbb{X},\mathbb{T}) \to VF(\mathbb{X},\mathbb{T})$

$M \mapsto (M, z_M)$

THE ZERO AND THE NEGATION

CONSTRUCTION

The zero:

$\mathcal{Z} : (F,\alpha) \Rightarrow \mathbb{T}(F,\alpha)$ is a vect. field:

$\mathcal{Z} \in VF[\mathbb{X}',\mathbb{T}' | \mathbb{X},\mathbb{T}]$

Take $(F,\alpha) = id_{(\mathbb{X},\mathbb{T})}$:

$\mathcal{Z} \in VF[\mathbb{X},\mathbb{T} | \mathbb{X},\mathbb{T}] \cong [\mathbb{X},\mathbb{T} | VF(\mathbb{X},\mathbb{T})]$

$\Rightarrow 0 : (\mathbb{X},\mathbb{T}) \to VF(\mathbb{X},\mathbb{T})$

$M \mapsto (M, z_M)$

The negation:

$\hat{v} : U \to \mathbb{T}U \in VF[VF(\mathbb{X},\mathbb{T}) | \mathbb{X},\mathbb{T}]$

THE ZERO AND THE NEGATION

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$$\Rightarrow - : VF(\mathbb{X},\mathbb{T}) \to VF(\mathbb{X},\mathbb{T})$$

$$(M,v) \mapsto (M,-v)$$

THE SUM AND THE LIE BRACKET

CONSTRUCTION

The sum:

$$\begin{tikzcd} VF_2(\mathbb{X},\mathbb{T}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & VF(\mathbb{X},\mathbb{T}) \arrow[d, "u"] \\ VF(\mathbb{X},\mathbb{T}) \arrow[r, "u"'] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

$$\pi_K \in [VF_2(\mathbb{X},\mathbb{T}) | VF(\mathbb{X},\mathbb{T})]$$

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$$(M,u,v) \mapsto (M,[u,v])$$

THE LIE ALGEBRA OF FORMAL VECTOR FIELDS

THEOREM

$u : VF(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$

$0 : (\mathbb{X}, \mathbb{T}) \to VF(\mathbb{X}, \mathbb{T})$

$- : VF(\mathbb{X}, \mathbb{T}) \to VF(\mathbb{X}, \mathbb{T})$

$+ : VF_2(\mathbb{X}, \mathbb{T}) \to VF(\mathbb{X}, \mathbb{T})$

$[,] : VF_2(\mathbb{X}, \mathbb{T}) \to VF(\mathbb{X}, \mathbb{T})$

is a Lie algebra object in

$TNG[\mathbb{K}] / (\mathbb{X}, \mathbb{T})$.

CONSTRUCTING FORMAL VECTOR FIELDS

THEOREM

When the ambient 2-category has (some) (P)IE limits every tangentad admits the construction of vector fields.

CONSTRUCTING FORMAL VECTOR FIELDS

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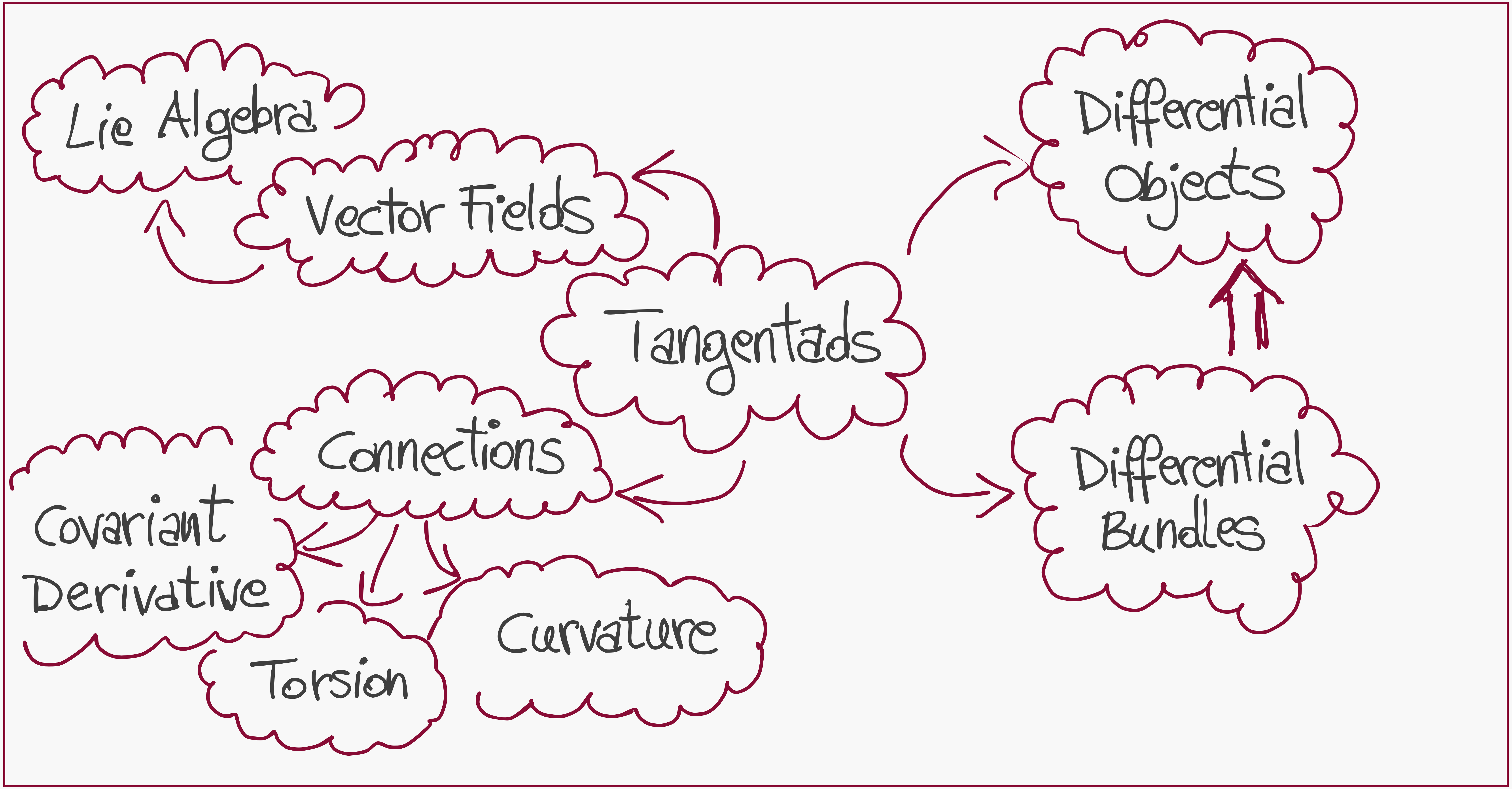
When the ambient 2-category has (some) (P)IE limits every tangentad admits the construction of vector fields.

In particular:
tangent monads,
tangent fibrations,
tangent (split) restr. cats
admit vector fields.

THE CONSTRUCTIONS THAT WE FORMALIZED



THANKS.

TANGENTADS: A FORMAL APPROACH TO TANGENT CATEGORIES
arxiv.org/abs/2503.18354

THE FORMAL THEORY OF TANGENTADS PART I
arxiv.org/abs/2509.15524

THE FORMAL THEORY OF TANGENTADS PART II
arxiv.org/abs/2601.15534