MACQUARIE UNIVERSITY

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REPRESENTABLE TANGENT STRUCTURES FOR AFFINE SCHEMES

JOINT WORK WITH JS LEMAY

MOTIVATION

AFFINE SCHEMES

COCKETT CRUTTWELL 2014 CRUTTWELL LEMAY 2023 CRUTTWELL LEMAY VANDENBERG 2024

TANGENT CATEGORY

Tangent bundle functor

$\mathbb{T} : \mathbb{X} \to \mathbb{X}$

ALGEBRAIC GEOMETRY

Fibré tangente (Grothendieck)

$\mathbb{T}A := \mathrm{Sym}_A \Omega_A$

MOTIVATION

AFFINE SCHEMES

COCKETT CRUTTWELL 2014
CRUTTWELL LEMAY 2023
CRUTTWELL LEMAY VANDENBERG 2024

TANGENT CATEGORY

Tangent bundle functor

$\mathbb{T} : \mathbb{X} \to \mathbb{X}$

Vector fields

$v : A \to \mathbb{T}A$

ALGEBRAIC GEOMETRY

Fibré tangente (Grothendieck)

$\mathbb{T}A := \mathrm{Sym}_A \Omega_A$

Derivations

$\delta : A \to A \quad \delta(ab) = a\delta(b) + b\delta(a)$

MOTIVATION

AFFINE SCHEMES

COCKETT CRUTTWELL 2014 — CRUTTWELL LEMAY 2023 — CRUTTWELL LEMAY VANDENBERG 2024

TANGENT CATEGORY	**ALGEBRAIC GEOMETRY**
Tangent bundle functor	Fibré tangente (Grothendieck)
$\mathbb{T}:\mathbb{X}\to\mathbb{X}$	$\mathbb{T}A := \mathrm{Sym}_A\Omega_A$
Vector fields	Derivations
$v:A\to\mathbb{T}A$	$\delta:A\to A \quad \delta(ab)=a\delta(b)+b\delta(a)$
Differential bundles	Modules
$q:E\to A$	$\mathrm{DB}_{\mathrm{LNR}}(\mathrm{AFF}_{R},\mathbb{T};A)\cong\mathrm{MOD}_A^{op}$

MOTIVATION

AFFINE SCHEMES

COCKETT CRUTTWELL 2014 — CRUTTWELL LEMAY 2023 — CRUTTWELL LEMAY VANDENBERG 2024

TANGENT CATEGORY	**ALGEBRAIC GEOMETRY**
Tangent bundle functor	Fibré tangente (Grothendieck)
$\mathbb{T} : \mathbb{X} \to \mathbb{X}$	$\mathbb{T}A := \mathrm{Sym}_A \Omega_A$
Vector fields	Derivations
$v : A \to \mathbb{T}A$	$\delta : A \to A \quad \delta(ab) = a\delta(b) + b\delta(a)$
Differential bundles	Modules
$q : E \to A$	$\mathsf{DB}_{\mathsf{LNR}}(\mathsf{AFF}_R, \mathbb{T}; A) \cong \mathsf{MOD}_A^{op}$
Connections on diff bundles	Connections on modules
$H : E \times_A \mathbb{T}A \to \mathbb{T}E$	$M \to \Omega_A \otimes_A M$

MOTIVATION

THE MAIN QUESTION

HOW SPECIAL IS THIS TANGENT STRUCTURE?

MOTIVATION

THE MAIN QUESTION

HOW SPECIAL IS	**THIS TANGENT STRUCTURE?**
HOW MANY OTHER TANGENT	**STRUCTURES ARE THERE ON AFF_R?**

MOTIVATION

THE MAIN QUESTION

THIS QUESTION IS TOO HARD TO ANSWER.

MOTIVATION

THE MAIN QUESTION

THIS QUESTION IS TOO HARD TO ANSWER.

HOW MANY REPRESENTABLE TANGENT STRUCTURES ON AFF_R?

MOTIVATION

THE PLAN FOR TODAY

REPRESENTABLE & INFINITESIMAL	TANGENT CATEGORIES OBJECTS

THE PLAN FOR TODAY

MOTIVATION

REPRESENTABLE TANGENT CATEGORIES
& INFINITESIMAL OBJECTS

TANGENTOIDS

MOTIVATION

THE PLAN FOR TODAY

REPRESENTABLE & INFINITESIMAL	TANGENT CATEGORIES OBJECTS
TANGEN	TOIDS
SYMMETRIC & COMMUTATIVE	TANGENTOIDS MONOIDS

MOTIVATION

THE PLAN FOR TODAY

REPRESENTABLE TANGENT CATEGORIES
& INFINITESIMAL OBJECTS

TANGENTOIDS

SYMMETRIC TANGENTOIDS
& COMMUTATIVE MONOIDS

CLASSIFICATION

CHAPTER 1

REPRESENTABLE TANGENT CATEGORIES

DEFINITION

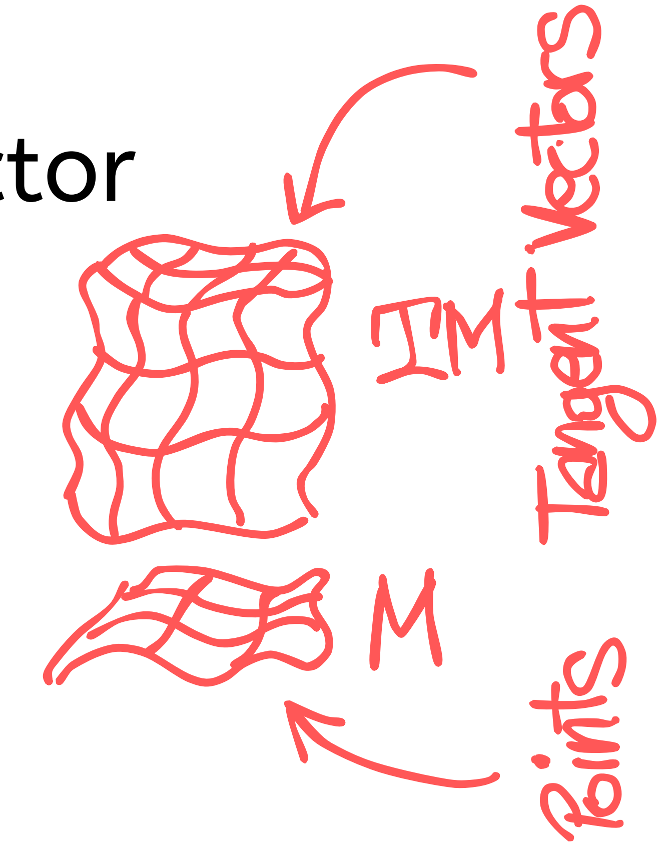
TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

TANGENT CATEGORY

Tangent bundle functor

$\mathbb{T} : \mathbb{X} \to \mathbb{X}$



DEFINITION

TANGENT CATEGORIES

ROSICKÝ 1984

COCKETT CRUTTWELL 2014

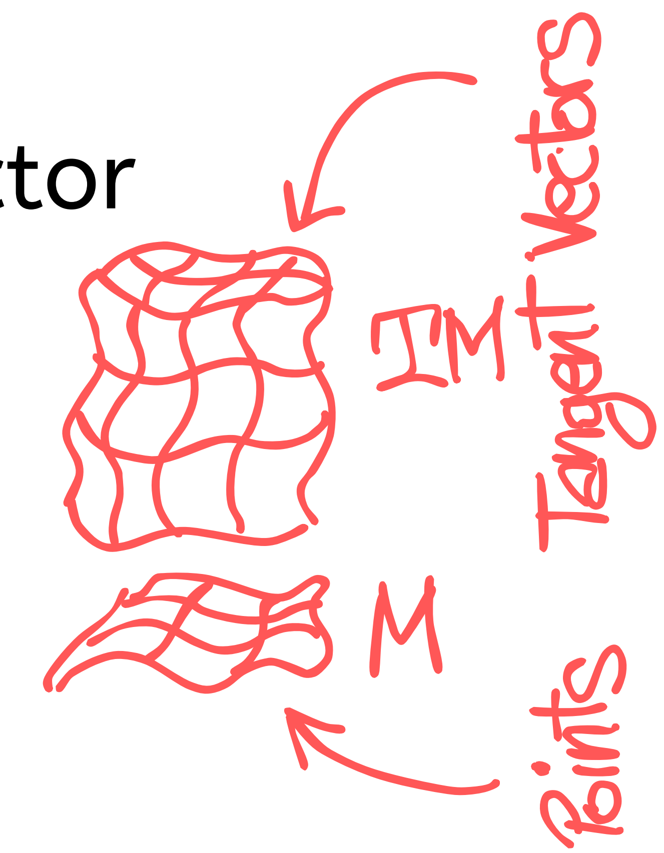
TANGENT CATEGORY

Tangent bundle functor

$$\mathbb{T} : \mathbb{X} \longrightarrow \mathbb{X}$$

Projection

$$p_M : \mathbb{T}M \longrightarrow M$$



TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

DEFINITION

TANGENT CATEGORY

Tangent bundle functor

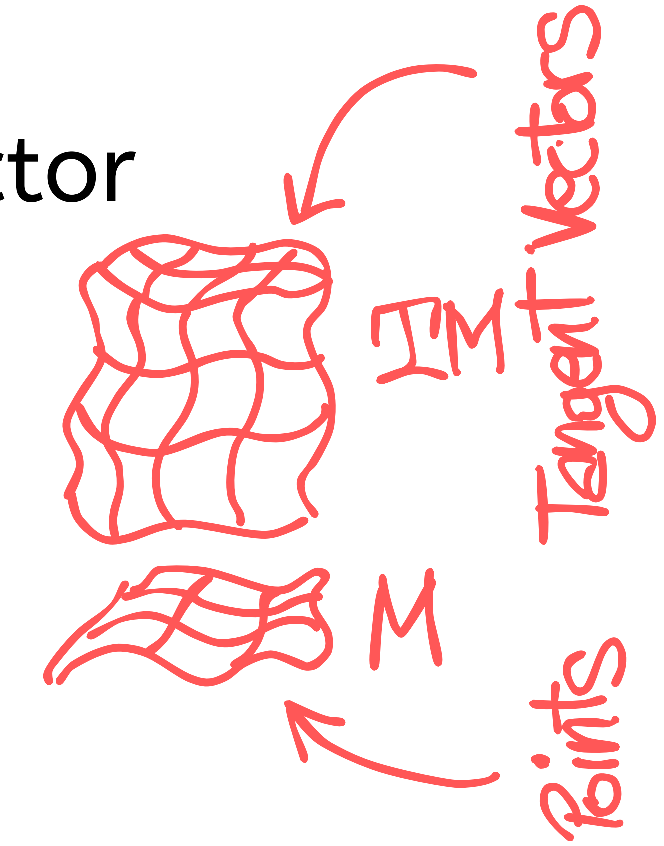
$$\mathbb{T} : \mathbb{X} \to \mathbb{X}$$

Projection

$$p_M : \mathbb{T}M \to M$$

Zero morphism

$$z_M : M \to \mathbb{T}M$$



DEFINITION

TANGENT CATEGORIES

ROSICKÝ 1984

COCKETT CRUTTWELL 2014

TANGENT CATEGORY

Tangent bundle functor

$$\mathbb{T}: \mathbb{X} \longrightarrow \mathbb{X}$$

Projection

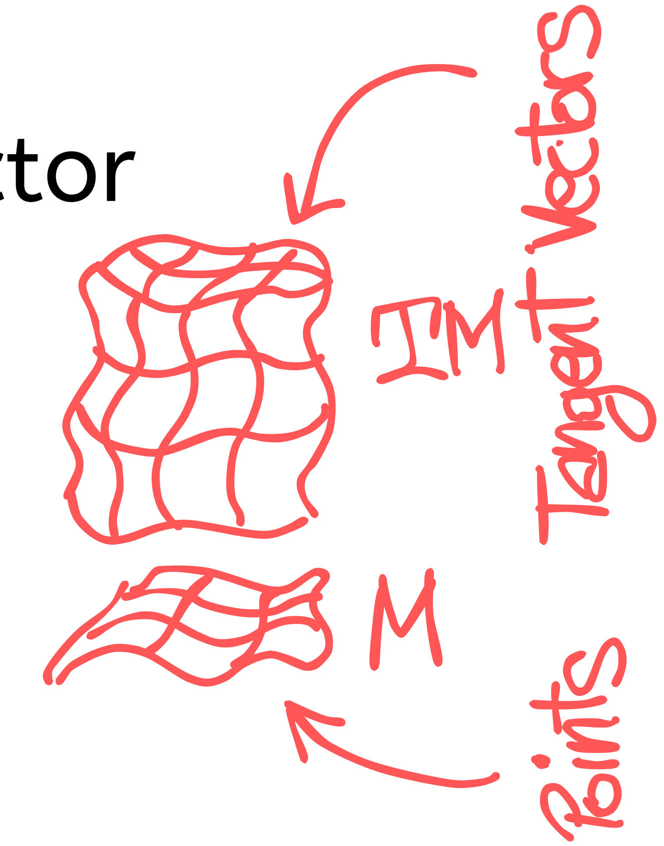
$$p_M: \mathbb{T}M \longrightarrow M$$

Zero morphism

$$z_M: M \longrightarrow \mathbb{T}M$$

Sum morphism

$$s_M: \mathbb{T}_2 M \longrightarrow \mathbb{T}M$$



DEFINITION

TANGENT CATEGORIES

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COCKETT CRUTTWELL 2014

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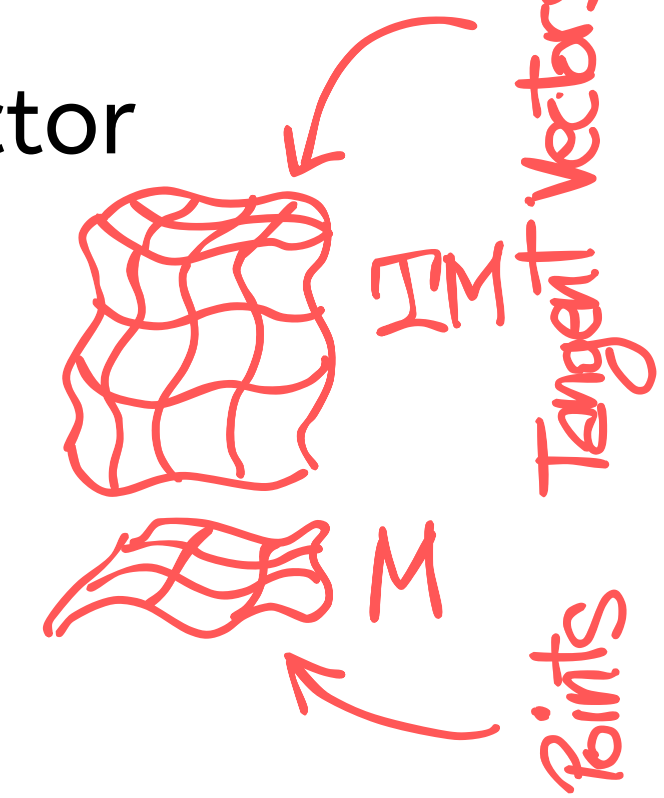


Additive structure on the tangent spaces

TANGENT CATEGORIES

ROSICKÝ 1984

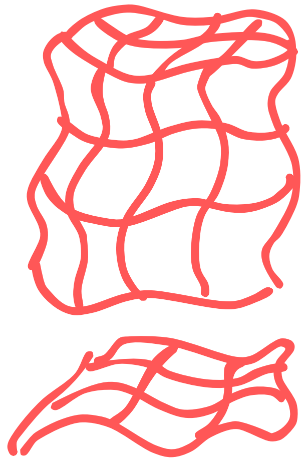
COCKETT CRUTTWELL 2014

DEFINITION

TANGENT CATEGORY

Tangent bundle functor

$$\mathbb{T} : \mathbb{X} \longrightarrow \mathbb{X}$$



Projection

$$p_M : \mathbb{T}M \longrightarrow M$$

Zero morphism

$$z_M : M \longrightarrow \mathbb{T}M$$

Sum morphism

$$s_M : \mathbb{T}_2M \longrightarrow \mathbb{T}M$$

Additive structure on the tangent spaces

Vertical lift

$$\ell_M : \mathbb{T}M \longrightarrow \mathbb{T}^2M$$

Local Linearity:

$$\mathbb{T}\mathbb{T}_xM = \mathbb{T}_xM \times \mathbb{T}_xM$$

DEFINITION

TANGENT CATEGORIES

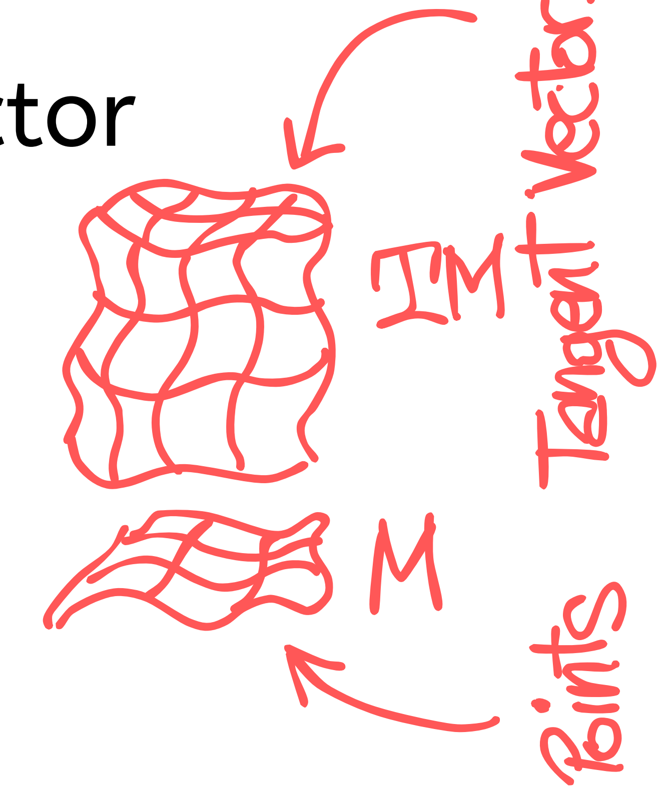
ROSICKÝ 1984

COCKETT
CRUTTWELL 2014

TANGENT CATEGORY

Tangent bundle functor

$T : \mathbb{X} \to \mathbb{X}$



Projection

$p_M : TM \to M$

Zero morphism

$z_M : M \to TM$

Sum morphism

$s_M : T_2M \to TM$

Additive structure on the tangent spaces

Vertical lift

$\ell_M : TM \to T^2M$

Local Linearity:

$TT_xM = T_xM \times T_xM$

Canonical flip

$c_M : T^2M \to T^2M$

Symmetry of partial derivatives

$\partial_x \partial_y = \partial_y \partial_x$

DEFINITION

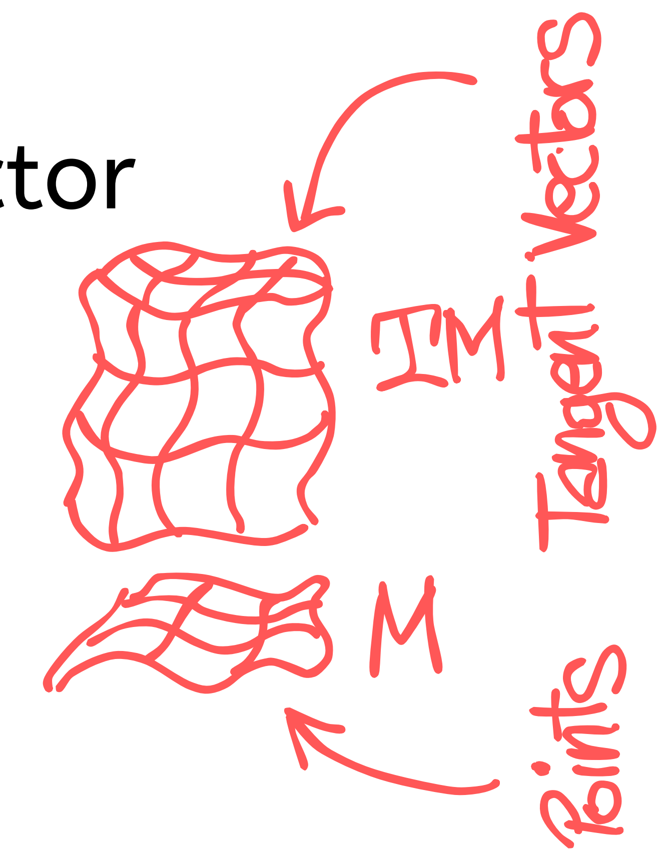
TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

TANGENT CATEGORY

Tangent bundle functor

$\mathbb{T} : \mathbb{X} \longrightarrow \mathbb{X}$



Projection

$p_M : \mathbb{T}M \longrightarrow M$

Zero morphism

$z_M : M \longrightarrow \mathbb{T}M$

Sum morphism

$s_M : \mathbb{T}_2 M \longrightarrow \mathbb{T}M$

Additive structure on the tangent spaces

Vertical lift

$\ell_M : \mathbb{T}M \longrightarrow \mathbb{T}^2 M$

Local Linearity:

$\mathbb{T}\mathbb{T}_x M = \mathbb{T}_x M \times \mathbb{T}_x M$

Canonical flip

$c_M : \mathbb{T}^2 M \longrightarrow \mathbb{T}^2 M$

Symmetry of partial derivatives

$\partial_x \partial_y = \partial_y \partial_x$

Negation

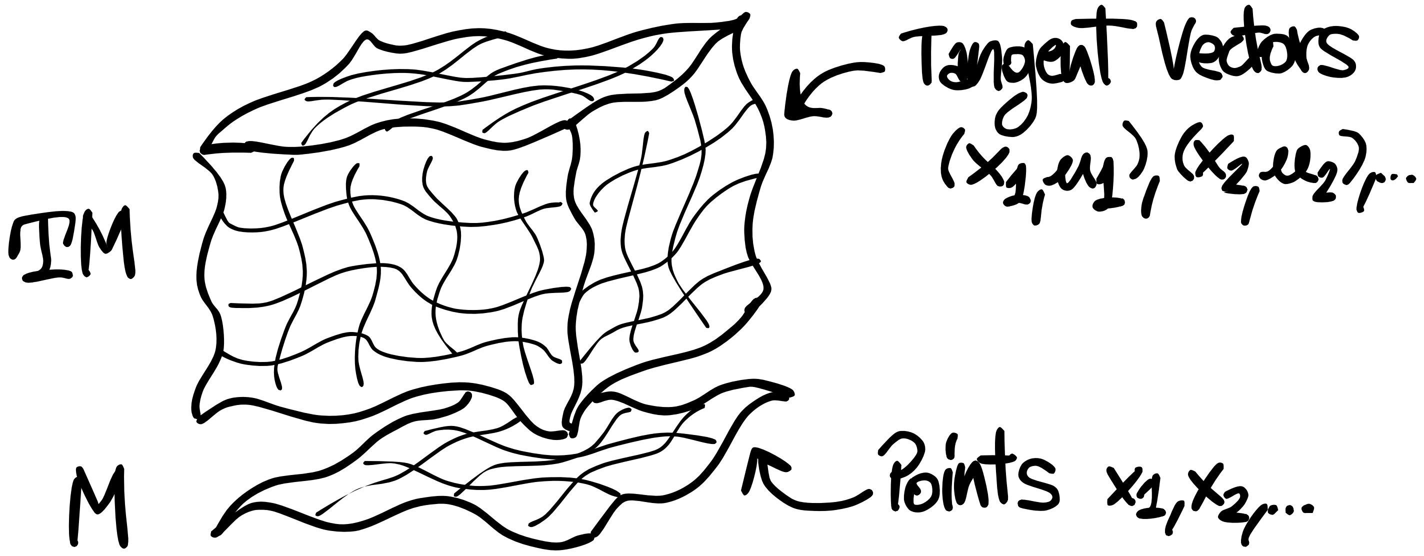
$n_M : \mathbb{T}M \longrightarrow \mathbb{T}M$

Abelian group structure on the fibres

EXAMPLES

TANGENT CATEGORIES

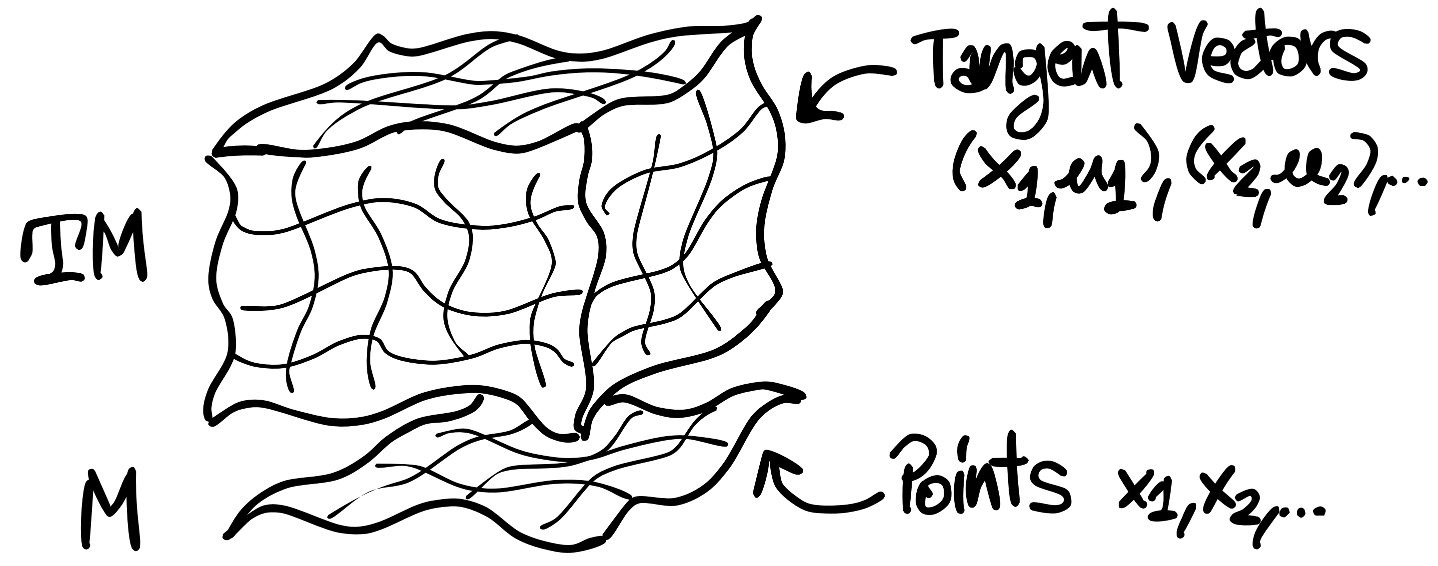
SMOOTH MANIFOLDS



EXAMPLES

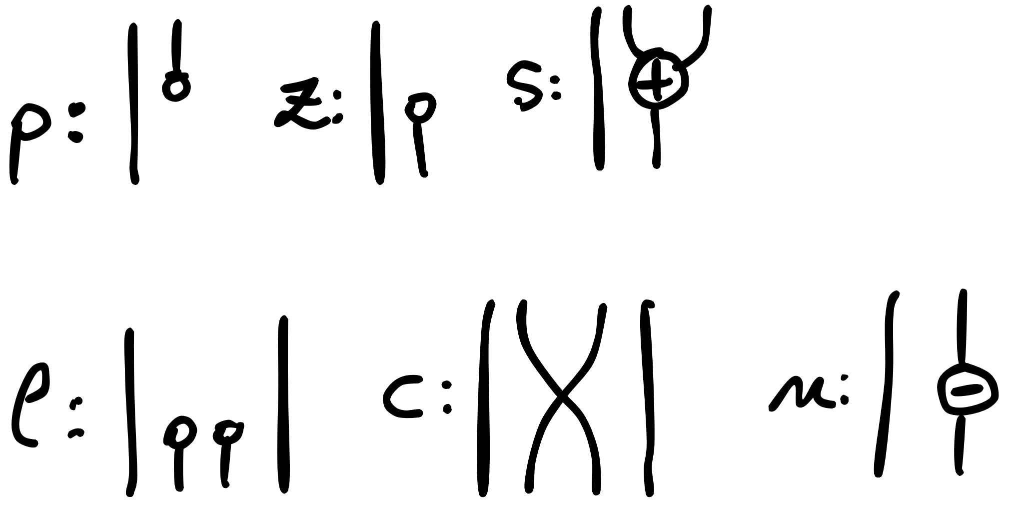
TANGENT CATEGORIES

SMOOTH MANIFOLDS



BIPRODUCTS

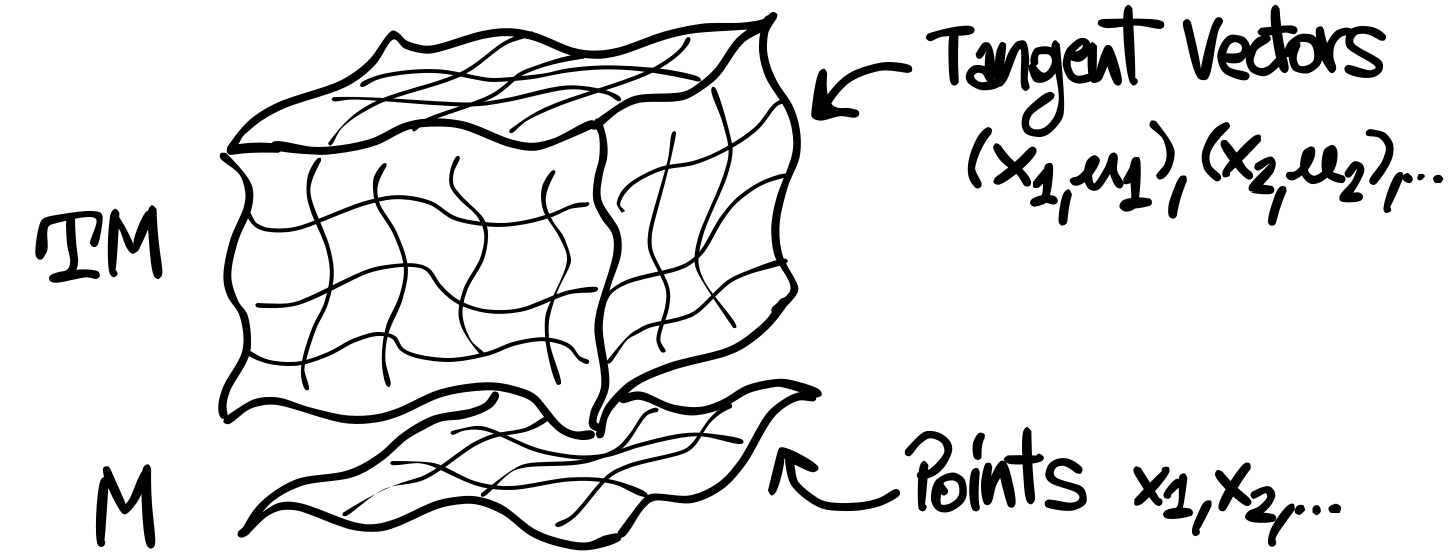
$$TM := M \oplus M$$



EXAMPLES

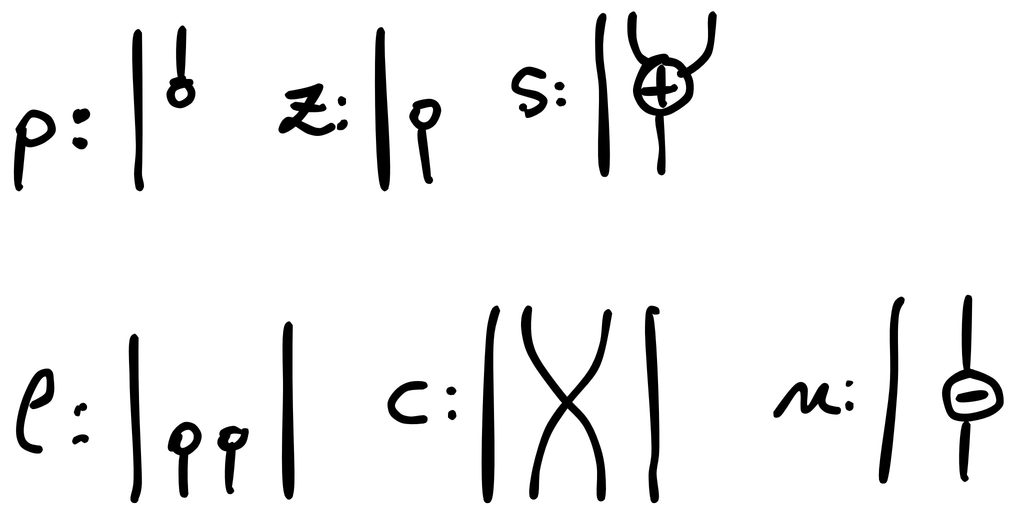
TANGENT CATEGORIES

SMOOTH MANIFOLDS



BIPRODUCTS

$$\mathbb{T}M := M \oplus M$$



COMMUTATIVE & UNITAL ALGEBRAS

$$\mathbb{T}A := R[\varepsilon]\otimes_R A$$

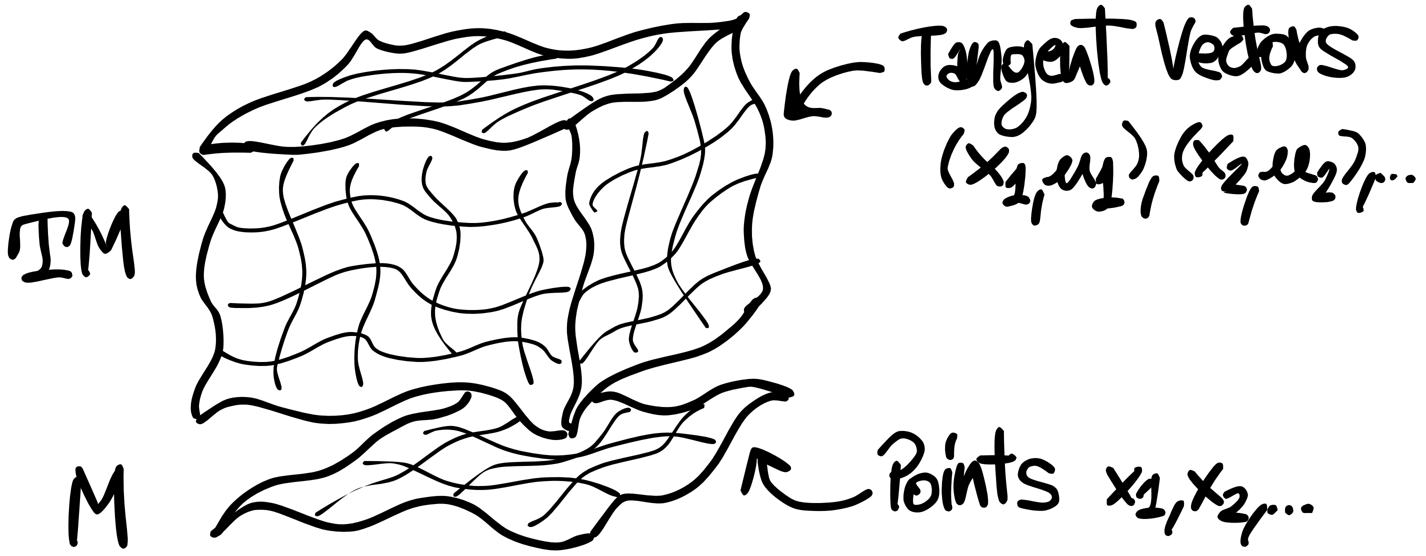
$a+b\varepsilon,\ \varepsilon^2=0$

$$R[\varepsilon] := \frac{R[x]}{(x^2)}$$

EXAMPLES

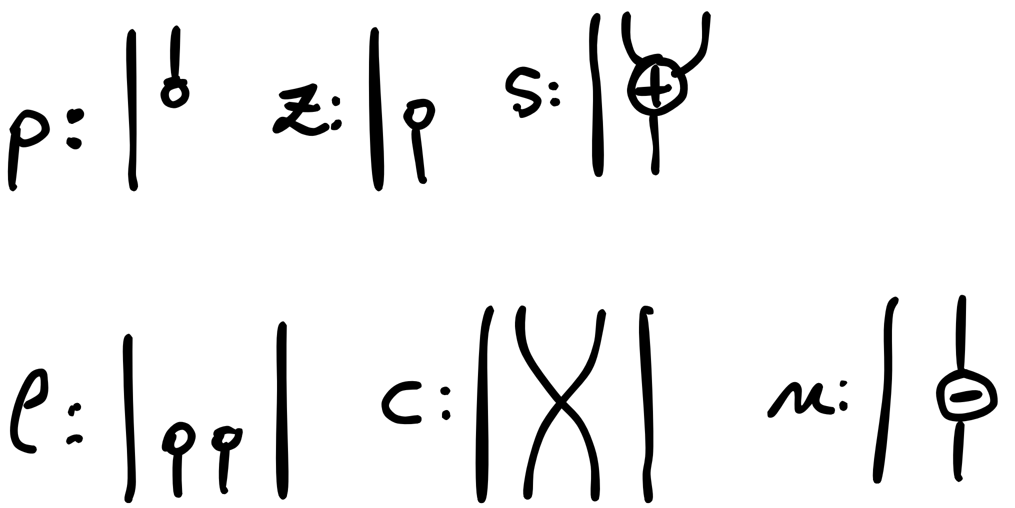
TANGENT CATEGORIES

SMOOTH MANIFOLDS



BIPRODUCTS

$TM := M \oplus M$



COMMUTATIVE & UNITAL ALGEBRAS

$TA := R[\varepsilon]\otimes_R A$

$a+b\varepsilon,\ \varepsilon^2=0$

$R[\varepsilon] := \dfrac{R[x]}{(x^2)}$

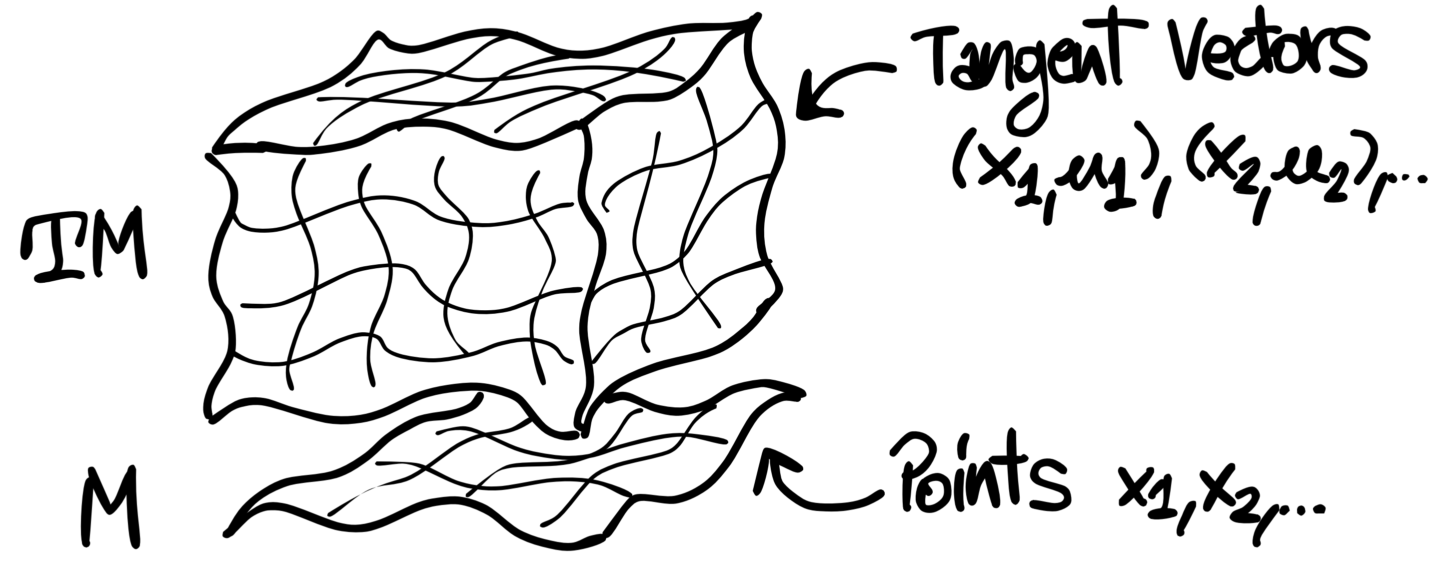
AFFINE SCHEMES

$TA = \mathrm{Sym}_A \Omega_A$

EXAMPLES

TANGENT CATEGORIES

SMOOTH MANIFOLDS



COMMUTATIVE & UNITAL ALGEBRAS

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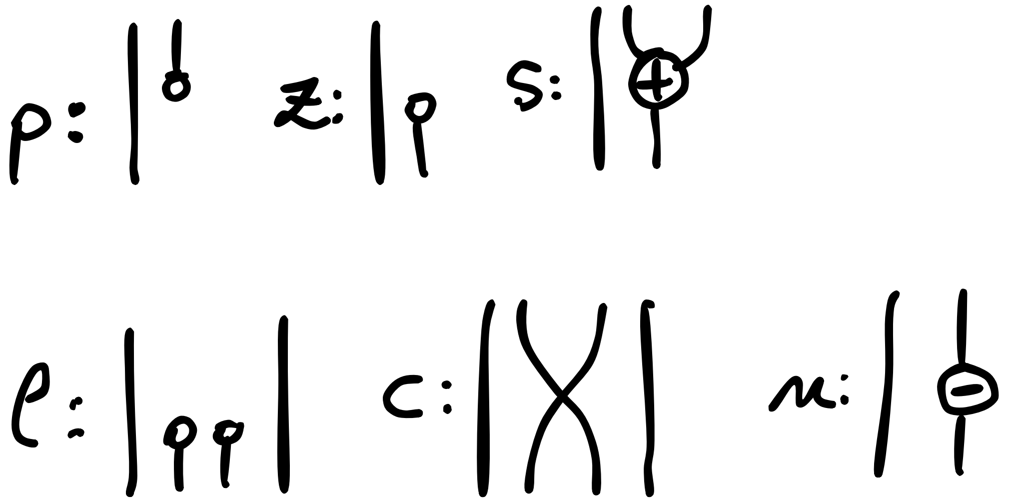
BIPRODUCTS

$TM := M \oplus M$



AFFINE SCHEMES

$TA = \mathrm{Sym}_A \Omega_A$

$a,\ da$ s.t.

$a \cdot_{TA} b = a \cdot_A b$

$d(ra + sb) = r\,da + s\,db$

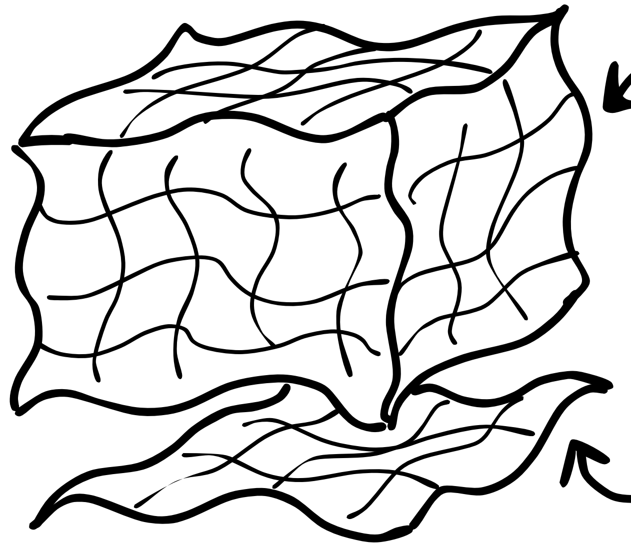
$d(ab) = a\,db + b\,da$

TANGENT CATEGORIES

SMOOTH MANIFOLDS

$\mathcal{T}M$

M



Tangent Vectors $(x_1,u_1),(x_2,u_2),\dots$

Points $x_1,x_2,\dots$

COMMUTATIVE & UNITAL ALGEBRAS

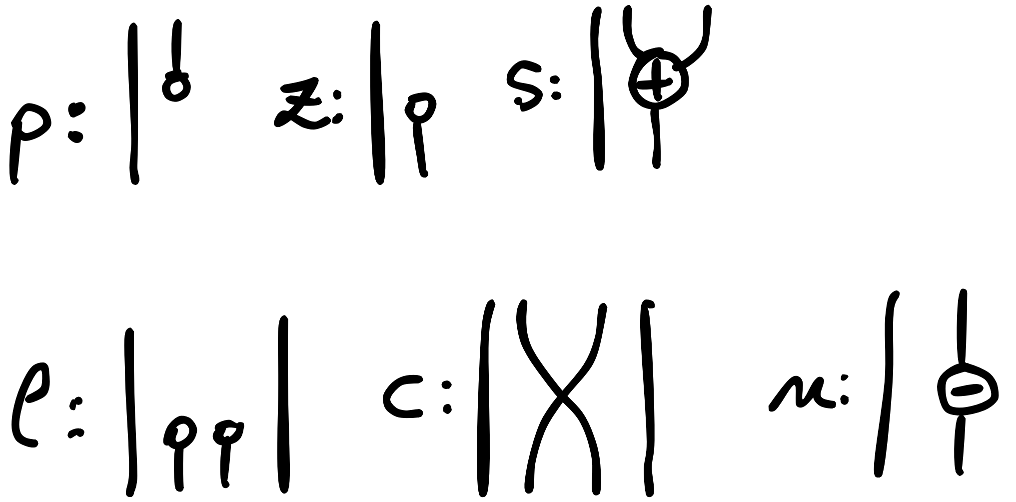
$\mathcal{T}A := R[\varepsilon]\otimes_R A$

$a+b\varepsilon,\ \varepsilon^2=0$

$R[\varepsilon] := \frac{R[x]}{(x^2)}$

BIPRODUCTS

$\mathcal{T}M := M\oplus M$



AFFINE SCHEMES

$\mathcal{T}A = \mathrm{Sym}_A\,\Omega_A$

$\parallel$

$A^{R[\varepsilon]}$

$a,\ da$ s.t.

$a\cdot_{\mathcal{T}A} b = a\cdot_A b$

$d(ra+sb) = r\,da + s\,db$

$d(ab) = a\,db + b\,da$

REPRESENTABLE TANGENT CATEGORIES

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DEFINITION

A Cartesian tangent category is **representable** when the tangent bundle functor

$$\mathbb{T} : \mathbb{X} \to \mathbb{X}$$

and each

$$\mathbb{T}_M : \mathbb{X} \to \mathbb{X}$$

are representable functors.

REPRESENTABLE TANGENT CATEGORIES

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DEFINITION

A Cartesian tangent category is **representable** when the tangent bundle functor

$$\mathbb{T} : \mathbb{X} \longrightarrow \mathbb{X}$$

and each

$$\mathbb{T}_M : \mathbb{X} \longrightarrow \mathbb{X}$$

$$\begin{tikzcd}
\mathbb{T}_M \arrow[r, "\pi_M"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{T} \arrow[d, "p"] \\
\mathbb{T} \arrow[r, "p"'] & \mathrm{id}_{\mathbb{X}}
\end{tikzcd}$$

are representable functors.

DEFINITION

REPRESENTABLE TANGENT CATEGORIES

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A Cartesian tangent category is **representable** when the tangent bundle functor

$$\mathbb{T} : \mathbb{X} \longrightarrow \mathbb{X}$$

and each

$$\mathbb{T}_n : \mathbb{X} \longrightarrow \mathbb{X}$$

$$\begin{tikzcd} \mathbb{T}_n \arrow[r, "\pi_n"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{T} \arrow[d, "p"] \\ \mathbb{T} \arrow[r, "p"'] & \mathrm{id}_{\mathbb{X}} \end{tikzcd}$$

are representable functors.

Concretely, there is a sequence

$$D, D_2, \ldots$$

of objects such that,
for each n>0, the functor

$$D_n \pitchfork - : \mathbb{X} \longrightarrow \mathbb{X}$$

is a left adjoint to the functor

$$\mathbb{T}_n : \mathbb{X} \longrightarrow \mathbb{X}$$

DEFINITION

INFINITESIMAL OBJECTS

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REPRESENTABLE TANGENT CAT	INFINITESIMAL OBJECT
Tangent bundle functor $(-)^D$	Object D

INFINITESIMAL OBJECTS

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DEFINITION

REPRESENTABLE TANGENT CAT

Tangent bundle functor $(-)^D$

Projection $p: M^D \to M$

INFINITESIMAL OBJECT

Object D

Projection $p: * \to D$

INFINITESIMAL OBJECTS

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DEFINITION

REPRESENTABLE TANGENT CAT	**INFINITESIMAL OBJECT**
Tangent bundle functor $(-)^D$	Object D
Projection $p : M^D \to M$	Projection $p : * \to D$
Zero morphism $z : M \to M^D$	Zero morphism $z : D \xrightarrow{!} *$

INFINITESIMAL OBJECTS

COCKETT
CRUTTWELL
2014

DEFINITION

REPRESENTABLE TANGENT CAT		**INFINITESIMAL OBJECT**	
Tangent bundle functor	$(-)^D$	Object	D
Projection	$p : M^D \to M$	Projection	$p : * \to D$
Zero morphism	$z : M \to M^D$	Zero morphism	$z : D \xrightarrow{!} *$
Sum morphism	$s : M^{D_2} \to M^D$	Sum morphism	$s : D \to D_2$

INFINITESIMAL OBJECTS

COCKETT
CRUTTWELL 2014

DEFINITION

REPRESENTABLE TANGENT CAT		INFINITESIMAL OBJECT	
Tangent bundle functor	$(-)^D$	Object	D
Projection	$p: M^D \to M$	Projection	$p: * \to D$
Zero morphism	$z: M \to M^D$	Zero morphism	$z: D \xrightarrow{!} *$
Sum morphism	$s: M^{D_2} \to M^D$	Sum morphism	$s: D \to D_2$
Vertical lift	$\ell: M^D \to M^{D \times D}$	Vertical lift	$\ell: D \times D \to D$

DEFINITION

INFINITESIMAL OBJECTS

COCKETT
CRUTTWELL
2014

REPRESENTABLE TANGENT CAT		INFINITESIMAL OBJECT	
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Sum morphism	$s: M^{D_2} \to M^D$	Sum morphism	$s: D \to D_2$
Vertical lift	$\ell: M^D \to M^{D\times D}$	Vertical lift	$\ell: D\times D \to D$
Canonical flip	$c: M^{D\times D} \to M^{D\times D}$	Canonical flip	$c: D\times D \xrightarrow{\tau} D\times D$

DEFINITION

INFINITESIMAL OBJECTS

COCKETT
CRUTTWELL 2014

REPRESENTABLE TANGENT CAT		INFINITESIMAL OBJECT	
Tangent bundle functor	$(-)^D$	Object	D
Projection	$p : M^D \to M$	Projection	$p : * \to D$
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Vertical lift	$\ell : M^D \to M^{D \times D}$	Vertical lift	$\ell : D \times D \to D$
Canonical flip	$c : M^{D \times D} \to M^{D \times D}$	Canonical flip	$c : D \times D \xrightarrow{\tau} D \times D$
Negation	$n : M^D \to M^D$	Negation	$n : D \to D$

DEFINITION

INFINITESIMAL OBJECTS

COCKETT
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2014

REPRESENTABLE TANGENT CAT

Tangent bundle functor $(-)^D$

Projection $p: M^D \to M$

Zero morphism $z: M \to M^D$

Sum morphism $s: M^{D_2} \to M^D$

Vertical lift $\ell: M^D \to M^{D \times D}$

Canonical flip $c: M^{D \times D} \to M^{D \times D}$

Negation $n: M^D \to M^D$

INFINITESIMAL OBJECT

Object D

Projection $p: * \to D$

Zero morphism $z: D \xrightarrow{!} *$

Sum morphism $s: D \to D_2$

Vertical lift $\ell: D \times D \to D$

Canonical flip $c: D \times D \xrightarrow{\tau} D \times D$

Negation $n: D \to D$

$D = R[\varepsilon]$

CHAPTER 2

TANGENTOIDS

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathbb{X}, \times, *)$

Object D

Projection $p : * \to D$

Zero morphism $z : D \xrightarrow{!} *$

Sum morphism $s : D \to D_2$

Vertical lift $\ell : D \times D \to D$

Canonical flip $c : D \times D \xrightarrow{\tau} D \times D$

Negation $n : D \to D$

TANGENTOID

Monoidal category $(\mathscr{C}, \otimes, I)$

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathbb{X}, \times, *)$

Object D

Projection $p: * \to D$

Zero morphism $z: D \xrightarrow{!} *$

Sum morphism $s: D \to D_2$

Vertical lift $\ell: D \times D \to D$

Canonical flip $c: D \times D \xrightarrow{\tau} D \times D$

Negation $n: D \to D$

TANGENTOID

Monoidal category $(\mathcal{C}, \otimes, I)$

$\mathbb{X}^{op}$

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathbb{X}, \times, *)$

Object D

Projection $p : * \to D$

Zero morphism $z : D \xrightarrow{!} *$

Sum morphism $s : D \to D_2$

Vertical lift $\ell : D \times D \to D$

Canonical flip $c : D \times D \xrightarrow{\tau} D \times D$

Negation $n : D \to D$

TANGENTOID

Monoidal category $(\mathcal{C}, \otimes, I)$

$\mathbb{X}^{op}$

Object D

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathbb{X}, \times, *)$

Object D

Projection $p: * \to D$

Zero morphism $z: D \xrightarrow{!} *$

Sum morphism $s: D \to D_2$

Vertical lift $\ell: D \times D \to D$

Canonical flip $c: D \times D \xrightarrow{\tau} D \times D$

Negation $n: D \to D$

TANGENTOID

Monoidal category $(\mathcal{C}, \otimes, I)$

$\mathbb{X}^{op}$

Object D

Projection $p: D \to I$

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathcal{X}, \times, *)$

Object D

Projection $p: * \to D$

Zero morphism $z: D \xrightarrow{!} *$

Sum morphism $s: D \to D_2$

Vertical lift $\ell: D \times D \to D$

Canonical flip $c: D \times D \xrightarrow{\tau} D \times D$

Negation $n: D \to D$

TANGENTOID

Monoidal category $(\mathcal{C}, \otimes, I)$ $\mathcal{X}^{op}$

Object D

Projection $p: D \to I$

Zero morphism $z: I \to D$

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT

Cartesian category $(\mathcal{X}, \times, *)$

Object D

Projection $p: * \longrightarrow D$

Zero morphism $z: D \xrightarrow{!} *$

Sum morphism $s: D \longrightarrow D_2$

Vertical lift $\ell: D \times D \longrightarrow D$

Canonical flip $c: D \times D \xrightarrow{\tau} D \times D$

Negation $n: D \longrightarrow D$

TANGENTOID

Monoidal category $(\mathcal{C}, \otimes, I)$

$\times_P$

Object D

Projection $p: D \longrightarrow I$

Zero morphism $z: I \longrightarrow D$

Sum morphism $s: D \times_I D \longrightarrow D$

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT		TANGENTOID	
Cartesian category	$(\mathcal{X}, \times, *)$	Monoidal category	$(\mathcal{C}, \otimes, I)$ $\otimes^{op}$
Object	D	Object	D
Projection	$p : * \to D$	Projection	$p : D \to I$
Zero morphism	$z : D \xrightarrow{!} *$	Zero morphism	$z : I \to D$
Sum morphism	$s : D \to D_2$	Sum morphism	$s : D \times_I D \to D$
Vertical lift	$\ell : D \times D \to D$	Vertical lift	$\ell : D \to D \otimes D$
Canonical flip	$c : D \times D \xrightarrow{\tau} D \times D$		
Negation	$n : D \to D$		

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT		**TANGENTOID**	
Cartesian category	$(\mathbb{X}, \times, *)$	Monoidal category	$(\mathcal{C}, \otimes, I)$
			$\mathbb{X}^{op}$
Object	D	Object	D
Projection	$p: * \to D$	Projection	$p: D \to I$
Zero morphism	$z: D \xrightarrow{!} *$	Zero morphism	$z: I \to D$
Sum morphism	$s: D \to D_2$	Sum morphism	$s: D \times_I D \to D$
Vertical lift	$\ell: D \times D \to D$	Vertical lift	$\ell: D \to D \otimes D$
Canonical flip	$c: D \times D \xrightarrow{\tau} D \times D$	Canonical flip	$c: D \otimes D \to D \otimes D$
Negation	$n: D \to D$		

DEFINITION

TANGENTOIDS

INFINITESIMAL OBJECT		TANGENTOID	
Cartesian category	$(\mathcal{X}, \times, *)$	Monoidal category	$(\mathcal{C}, \otimes, I)$ $\mathcal{X}^{op}$
Object	D	Object	D
Projection	$p : * \to D$	Projection	$p : D \to I$
Zero morphism	$z : D \xrightarrow{!} *$	Zero morphism	$z : I \to D$
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Vertical lift	$\ell : D \times D \to D$	Vertical lift	$\ell : D \to D \otimes D$
Canonical flip	$c : D \times D \xrightarrow{\tau} D \times D$	Canonical flip	$c : D \otimes D \to D \otimes D$
Negation	$n : D \to D$	Negation	$n : D \to D$

DEFINITION

SYMMETRIC TANGENTOIDS

INFINITESIMAL OBJECT	**SYMMETRIC TANGENTOID**
Cartesian category $(\mathcal{X}, \times, *)$	Symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$
Object D	Object D
Projection $p : * \to D$	Projection $p : D \to I$
Zero morphism $z : D \xrightarrow{!} *$	Zero morphism $z : I \to D$
Sum morphism $s : D \to D_2$	Sum morphism $s : D \times_I D \to D$
Vertical lift $\ell : D \times D \to D$	Vertical lift $\ell : D \to D \otimes D$
Canonical flip $c : D \times D \xrightarrow{\sim} D \times D$	Canonical flip $c : D \otimes D \to D \otimes D = \sigma_{D,D}$
Negation $n : D \to D$	Negation $n : D \to D$

EXAMPLES

SYMMETRIC TANGENTOIDS

TRIVIAL TANGENT STRUCTURE

In every $(\mathcal{E}, \otimes, I)$, I is a tangentoid

EXAMPLES

SYMMETRIC TANGENTOIDS

TRIVIAL TANGENT STRUCTURE

In every $(\mathcal{C}, \otimes, I)$, I is a tangentoid

MODULES

In MOD_R, $I \oplus I$ is a tangentoid and

$$\mathbb{T}M = (I \oplus I) \otimes M = M \oplus M$$

EXAMPLES

SYMMETRIC TANGENTOIDS

TRIVIAL TANGENT STRUCTURE

In every $(\mathcal{E}, \otimes, I)$, I is a tangentoid

MODULES

In MOD_R, $I \oplus I$ is a tangentoid and

$$TM = (I \oplus I) \otimes M = M \oplus M$$

COMMUTATIVE & UNITAL ALGEBRAS

In CALG_R, $R[\varepsilon]$ is a tangentoid

$$TA = R[\varepsilon] \otimes A$$

EXAMPLES

SYMMETRIC TANGENTOIDS

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In CALG_R, $R[\varepsilon]$ is a tangentoid

$$\mathrm{T}A = R[\varepsilon] \otimes A$$

MODULES

In MOD_R, $I \oplus I$ is a tangentoid and

$$\mathrm{T}M = (I \oplus I) \otimes M = M \oplus M$$

AFFINE SCHEMES

Only the trivial one!

PROPOSITION

TANGENTOIDS

D is a tangentoid in a monoidal category

$$(\mathscr{C}, \otimes, I)$$

if and only if

$$\mathbb{T} := D \otimes -$$

$$p := p^D \otimes - \quad z := z^D \otimes - \quad s := s^D \otimes -$$

$$\ell := \ell^D \otimes - \quad c := c^D \otimes - \quad m := m^D \otimes -$$

is a tangent structure.

COEXPONENTIABLE TANGENTOIDS

DEFINITION

A **coexponentiable** tangentoid in a monoidal category

$$(\mathcal{C}, \otimes, I)$$

is a tangentoid for which each

$$D_n \otimes - : \mathcal{C} \to \mathcal{C}$$

admits a left adjoint.

COEXPONENTIABLE TANGENTOIDS

PROPOSITION

In a coCartesian monoidal category every tangentoid is symmetric.

PROPOSITION

COEXPONENTIABLE TANGENTOIDS

In a coCartesian monoidal category every tangentoid is symmetric.

Moreover, TFAE:

D is a coexponentiable tangentoid

D infinitesimal object in the opposite category

$(-)^D$ representable tangent structure on the opposite category

PROPOSITION

COEXPONENTIABLE TANGENTOIDS

In a coCartesian monoidal category every tangentoid is symmetric.

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D is a coexponentiable tangentoid

D infinitesimal object in the opposite category

$(-)^D$ representable tangent structure on the opposite category

In particular, there is a correspondence between:

representable tangent structures on AFF_R

coexponentiable tangentoids in CALG_R

CHAPTER 3

SYMMETRIC TANGENTOIDS & COMMUTATIVE MONOIDS

LEMMA

THE MONOID OF A TANGENTOID

Every tangentoid D comes with a monoid structure

$$\eta = z : I \longrightarrow D$$

$$\mu = D \otimes D \xrightarrow{\langle D \otimes p, p \otimes D \rangle} D \times_I D \xrightarrow{s} D$$

Moreover, in a symmetric monoidal category, such a monoid is always commutative.

LEMMA

THE MONOID OF A TANGENTOID

Every tangentoid D comes with a monoid structure

$$\eta = z : I \longrightarrow D$$

$$\mu = D \otimes D \xrightarrow{\langle D \otimes p_1, p \otimes D \rangle} D \times_I D \xrightarrow{s} D$$

Moreover, in a symmetric monoidal category, such a monoid is always commutative.

We can define a functor:

$$\mathrm{TOID}(\mathscr{C}, \otimes, I) \longrightarrow \mathrm{CMON}(\mathscr{C}, \otimes, I)$$

PROPOSITION

THE MONOID OF A TANGENTOID

In a symmetric monoidal category the functor

$$\mathrm{TOID}(\mathscr{C}, \otimes, I) \longrightarrow \mathrm{CMON}(\mathscr{C}, \otimes, I)$$

extends to a functor

$$\mathrm{STOID}(\mathscr{C}, \otimes, I, \sigma) \to (\mathrm{S})\mathrm{TOID}(\mathrm{CMON}(\mathscr{C}, \otimes, I, \sigma))$$

which is an isomorphism of categories, with inverse the forgetful functor

$$\mathrm{STOID}(U : \mathrm{CMON}(\mathscr{C}) \to \mathscr{C})$$

PROPOSITION

THE MONOID OF A TANGENTOID

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$$\mathrm{TOID}(\mathscr{C},\otimes,I) \longrightarrow \mathrm{CMON}(\mathscr{C},\otimes,I)$$

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which is an isomorphism of categories, with inverse the forgetful functor

$$\mathrm{STOID}(U : \mathrm{CMON}(\mathscr{C}) \to \mathscr{C})$$

In particular:

$$\mathrm{STOID}(\mathrm{MOD}_R) \cong \mathrm{STOID}(\mathrm{CALG}_R)$$

CHAPTER 4

CLASSIFICATION

SYMMETRIC TANGENTOIDS IN MOD_R

IN MOD_R

A symmetric tangentoid in MOD_R comprises:

Object D

Projection $p: D \longrightarrow R$

Zero morphism $z: R \longrightarrow D$

Sum morphism $s: D \times_R D \longrightarrow D$

Vertical lift $\ell: D \longrightarrow D \otimes D$

SYMMETRIC TANGENTOIDS IN MOD_R

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From the additive structure:

$D \cong R \oplus M, \quad M := \ker p$

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Vertical lift $\ell: D \to D \otimes D$

From the additive structure:

$D \cong R \oplus M, \quad M := \ker p$

$p: R \oplus M \xrightarrow{\pi_R} R$

$z: R \xrightarrow{\mathrm{id}_R \oplus 0} R \oplus M$

$s: R \oplus M \oplus M \xrightarrow{\mathrm{id}_R \oplus +} R \oplus M$

SYMMETRIC TANGENTOIDS IN MOD$_R$

IN MOD$_R$

A symmetric tangentoid in MOD$_R$ comprises:

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Projection $p : D \to R$

Zero morphism $z : R \to D$

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$n : R \oplus M \xrightarrow{id_R \oplus -} R \oplus M$

SYMMETRIC TANGENTOIDS IN MOD_R

IN MOD_R

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$s: R \oplus M \oplus M \xrightarrow{id_R \oplus +} R \oplus M$

$n: R \oplus M \xrightarrow{id_R \oplus -} R \oplus M$

From the vertical lift:

$u: M \to R \oplus M \xrightarrow{\ell} (R \oplus M) \otimes (R \oplus M) \xrightarrow{\pi_M \otimes \pi_M} M \otimes M$

LEMMA

SYMMETRIC TANGENTOIDS IN MOD_R

The R-module M with the map

$$\nu: M \to R \oplus M \xrightarrow{\rho} (R \oplus M) \otimes (R \oplus M) \xrightarrow{\pi_M \otimes \pi_M} M \otimes M$$

forms a cocommutative non-unital coalgebra.

SYMMETRIC TANGENTOIDS IN MOD$_R$

LEMMA

The R-module M with the map

$$u : M \to R \oplus M \xrightarrow{e} (R \oplus M) \otimes (R \oplus M) \xrightarrow{\pi_M \otimes \pi_M} M \otimes M$$

forms a cocommutative non-unital coalgebra.

Moreover, the following diagram commutes:

$$\begin{tikzcd} R \oplus M \arrow[r, "id_R \oplus u"] \arrow[dr, "e"'] & R \oplus (M \otimes M) \arrow[d, "e^\#"] \\ & (R \oplus M) \otimes (R \oplus M) \end{tikzcd}$$

SYMMETRIC TANGENTOIDS IN MOD_R

LEMMA

The R-module M with the map

$$u : M \to R\oplus M \xrightarrow{e} (R\oplus M)\otimes(R\oplus M) \xrightarrow{\pi_M\otimes\pi_M} M\otimes M$$

forms a cocommutative non-unital coalgebra.

Moreover, the following diagram commutes:

$$\begin{tikzcd}
R\oplus M \arrow[r, "\mathrm{id}_R\oplus u"] \arrow[dr, "e"'] & R\oplus(M\otimes M) \arrow[d, "e^{\#}"] \\
 & (R\oplus M)\otimes(R\oplus M)
\end{tikzcd}$$

Where:

$$\begin{tikzcd}
R\oplus(M\otimes M) \arrow[r, "e^{\#}"] & R\oplus M\oplus M\oplus(M\otimes M) \arrow[d, equal] \\
 & (R\oplus M)\otimes(R\oplus M)
\end{tikzcd}$$

SYMMETRIC TANGENTOIDS IN MOD$_R$

UNIVERSALITY

The universality of the vertical lift establishes that the following diagram must be a triple equalizer:

$$\begin{tikzcd} D \arrow[r, "e"] & D\otimes D \arrow[r, bend left, "D\otimes p"] \arrow[r, "p\otimes D"] \arrow[r, bend right, "pz\otimes p"'] & D \end{tikzcd}$$

UNIVERSALITY

SYMMETRIC TANGENTOIDS IN MOD$_R$

The universality of the vertical lift establishes that the following diagram must be a triple equalizer:

$$
\begin{tikzcd}
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R \oplus (M \otimes M) \arrow[u, "e^{\#}"'] & &
\end{tikzcd}
$$

UNIVERSALITY

SYMMETRIC TANGENTOIDS IN MOD$_R$

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$$
\begin{tikzcd}
D \arrow[r, "e"] & D\otimes D \arrow[r, bend left, "D\otimes p"] \arrow[r, "p\otimes D" description] \arrow[r, bend right, "pz\otimes p"'] & D \\
R\oplus(M\otimes M) \arrow[u, dashed, "e^0"] \arrow[ur, "e^{\#}"'] & &
\end{tikzcd}
$$

UNIVERSALITY

SYMMETRIC TANGENTOIDS IN MOD$_R$

The universality of the vertical lift establishes that the following diagram must be a triple equalizer:

$$
\begin{tikzcd}
D \arrow[r, "e"] & D\otimes D \arrow[r, bend left, "D\otimes p"] \arrow[r, "p\otimes D"] \arrow[r, bend right, "p_Z\otimes p"'] & D \\
R\oplus(M\otimes M) \arrow[u, dashed, "e^{\circ}"] \arrow[ur, "e^{\#}"'] & &
\end{tikzcd}
$$

$$\alpha : M\otimes M \xrightarrow{\iota_{M\otimes M}} R\oplus(M\otimes M) \xrightarrow{e^{\circ}} R\oplus M \xrightarrow{\pi_M} M$$

PROPOSITION

SYMMETRIC TANGENTOIDS IN MOD_R

The universality of the vertical lift establishes that the following diagram must be a triple equalizer:

$$\begin{tikzcd}
D \arrow[r, "e"] & D\otimes D \arrow[r, bend left, "D\otimes p"] \arrow[r, "p\otimes D"] \arrow[r, bend right, "pz\otimes p"'] & D \\
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\end{tikzcd}$$

$$\alpha : M\otimes M \xrightarrow{\iota_{M\otimes M}} R\oplus(M\otimes M) \xrightarrow{e^{0}} R\oplus M \xrightarrow{\pi_M} M$$

The R-module M associated with a symmetric tangentoid D of MOD_R is a **solid** commutative non-unital R-algebra, namely, it is a commutative non-unital R-algebra

$$M,\ \alpha : M\otimes M \to M$$

whose multiplication is invertible

$$\alpha : M\otimes M \rightleftarrows M : \upsilon$$

THEOREM

SYMMETRIC TANGENTOIDS IN MOD_R

The R-module M associated with a symmetric tangentoid D of MOD_R is a solid commutative non-unital R-algebra

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THEOREM

SYMMETRIC TANGENTOIDS IN MOD_R

The R-module M associated with a symmetric tangentoid D of MOD_R is a solid commutative non-unital R-algebra

$$M, \alpha : M \otimes M \to M$$

Every solid commutative non-unital R-algebra defines a symmetric tangentoid of MOD_R:

$$D = R \oplus M$$

THEOREM

SYMMETRIC TANGENTOIDS IN CALG_R

The R-module M associated with a symmetric tangentoid D of MOD_R is a solid commutative non-unital R-algebra

$$M, \ \alpha : M \otimes M \to M$$

Every solid commutative non-unital R-algebra defines a symmetric tangentoid of MOD_R:

$$D = R \oplus M$$

The R-module M associated with a symmetric tangentoid D of CALG_R is a solid commutative non-unital R-algebra

$$M, \ \alpha : M \otimes M \to M$$

Every solid commutative non-unital R-algebra defines a symmetric tangentoid of CALG_R:

$$D = R \ltimes M$$

THEOREM

SYMMETRIC TANGENTOIDS IN $CALG_R$

The R-module M associated with a symmetric tangentoid D of MOD_R is a solid commutative non-unital R-algebra

$$M, \alpha : M \otimes M \to M$$

Every solid commutative non-unital R-algebra defines a symmetric tangentoid of MOD_R:

$$D = R \oplus M$$

The R-module M associated with a symmetric tangentoid D of $CALG_R$ is a solid commutative non-unital R-algebra

$$M, \alpha : M \otimes M \to M$$

Every solid commutative non-unital R-algebra defines a symmetric tangentoid of $CALG_R$:

$$D = R \ltimes M$$

$r + x\varepsilon, \ \varepsilon^2 = 0$

EXAMPLES

A NEW TANGENT STRUCTURE!

The ring Q of rational numbers define a symmetric tangentoid

$$D^{\mathbb{Q}} = \mathbb{Z} \ltimes \mathbb{Q}$$

in CRING.
Therefore, it defines a new tangent structure on CRING:

$$T^{\mathbb{Q}}A = \mathbb{Z} \ltimes (\mathbb{Q} \otimes A)$$

EXPONENTIABLE AFFINE SCHEMES

NIEFIELD 1982

THEOREM

An affine scheme A is exponentiable if and only if its underlying R-module is finitely generated and projective.

THEOREM

COEXPONENTIABLE TANGENTOIDS IN CALG_R

There is a correspondence between:

THEOREM

COEXPONENTIABLE TANGENTOIDS IN CALG$_R$

There is a correspondence between:

Infinitesimal objects D of AFF$_R$

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COEXPONENTIABLE TANGENTOIDS IN CALG_R

There is a correspondence between:

Infinitesimal objects D of AFF_R

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Coexponentiable tangentoids D of CALG_R

Finitely generated, projective R-modules M with a structure of commutative solid non-unital algebras

COROLLARY

COEXPONENTIABLE TANGENTOIDS IN $CALG_R$

When the base ring R is a PID, the only coexponentiable tangentoids of $CALG_R$ are

$$R = R \ltimes 0 \qquad R[\varepsilon] = R \ltimes R$$

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When the base ring R is a PID, the only coexponentiable tangentoids of CALG_R are

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Therefore, when R is a PID, there are exactly two representable tangent structures on AFF_R:

the trivial one $\quad \mathbb{T}A = A$

the one of Kahler differentials

$$\mathbb{T}A = \mathrm{Sym}_A \Omega_A$$

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In particular, the category AFF of affine schemes over Z has exactly two representable tangent structures.

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Proof. M finitely gen. & projective.

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$$R^n \cong M \cong M \otimes M \cong R^{n^2}$$

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Proof. M finitely gen. & projective.

But R PID: projective $\Leftrightarrow$ free

$\Rightarrow M = R^n,\ n \in \mathbb{N}.$

$$R^n \cong M \cong M \otimes M \cong R^{n^2}$$

$$\Rightarrow n = n^2 \Rightarrow n \begin{smallmatrix} \nearrow 0 \\ \searrow 1 \end{smallmatrix} \Rightarrow M \begin{smallmatrix} \nearrow 0 \\ \searrow R \end{smallmatrix}$$

A NEW REPRESENTABLE TANGENT STRUCTURE!

EXAMPLES

Take a unital and commutative ring S and let

$$R = S \times S$$

Then R is not a PID because it has non-zero divisors.

However, S is finitely generated since

$$S = R/(1,0)$$

and projective, since SxS is free. Moreover, S is solid:

$$S \otimes S \cong S$$

EXAMPLES

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$$S \otimes S \cong S$$

Thus, S defines a coexponentiable tangentoid of CALG_R and a representable tangent structure on AFF_R.

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THANKS FOR
LISTENING