Tangent Categories
A MINI COURSE

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DISCLAIMER. These notes want to be an informal introduction to tangent categories and should not be regarded as a complete or fully detailed textbook. Here, I put my personal point of view and interpetations on tangent categories, point of view that I maturated during my Ph.D. Other people in the tangent community might have different perspectives, which I believe to be as valid as mine. I am not an expert in differential geometry nor algebraic geometry and I am firmly convinced that tangent categories do not and should not try to replace either of them. Finally, I am not an English native speaker: some sentences might be a bit idiosyncratic. I did my best (within the limited time I had) to make these notes simple and readable, accessible to people from different backgrounds. Any positive suggestions for improvement is very welcome.

Contents

1 The tangent community

[Jiří Rosický]: [Ros84]

[Robin Cockett]: [CC14; CC18; CC17; CLL20; CS25; CCL21]

[Geoffrey Cruttwell]: [CC14; CC18; CC17; CL25; CL23; CLV25]. Some really nice notes can be found here

[Rory Lucyshyn-Wright]: [Luc18]

[Kristine Bauer]: [BBC21]

[Lory Aintablian]: [AB25]

[Christian Blohman]: [AB25]

[Jonathan Gallagher]: His Ph.D. thesis can be found here

[Dorette Pronk]: [DV23]

[Michael Ching]: [BBC21; Chi24; Chi21]

[Marcello Lanfranchi]: [ILL24; Lan24; Lan25c; CL25; Lan25a; Lan25b; Lan26; LL25]

[Geoff Vooys]: [DV23; LV25; Voo23]

[Tim Van Der Linden]: [SLV25]

[Poon Leung]: [Leu17]

[Rick Blute]: [BCL19; BCS06; BCS09]

[Richard Garner]: [Gar18]

[JS Lemay]: [CL23; Lem19; CCL21; SLV25; ILL24; LL25]

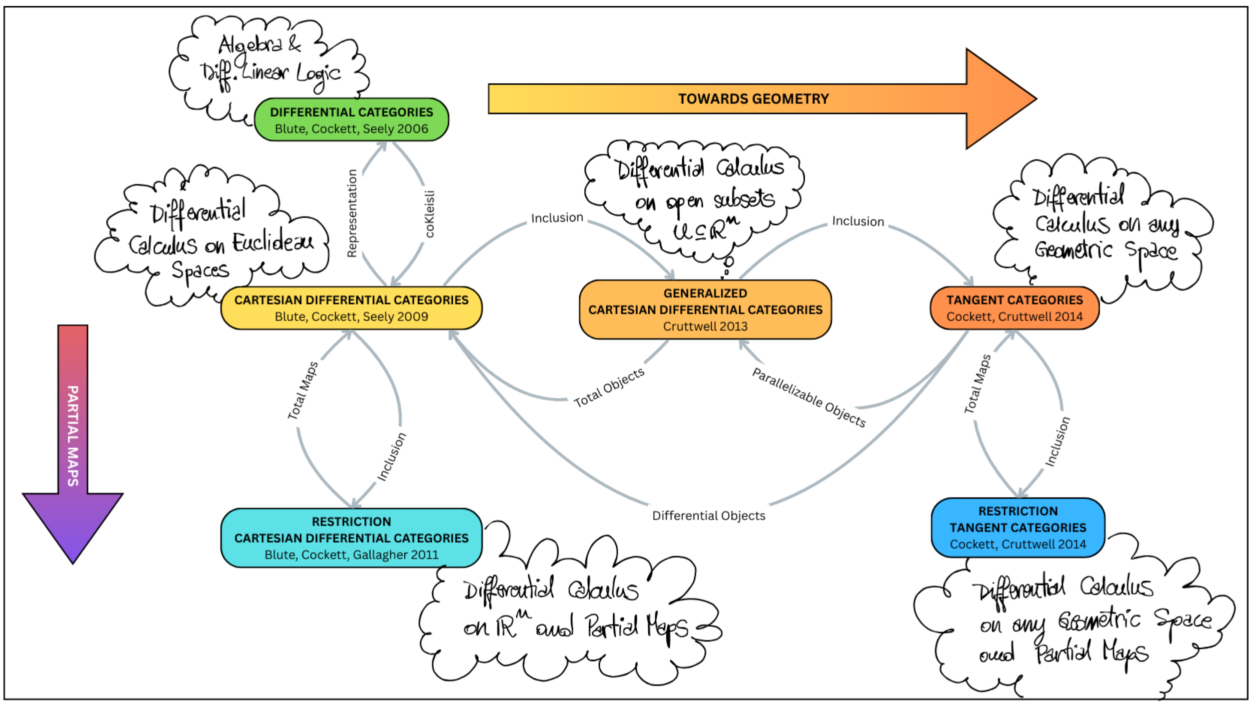
[Florian Schwarz]: [CS25; Sch26]

[Ben MacAdam]: [Mac22; Mac21]

[Sacha Ikonicoff]: [SLV25; ILL24; IL21]

2 The world of differential categories

Tangent categories are only one of the creatures that populate the world of differential categories. The beasts of this world provide categorical frameworks for concepts related to differentiation and differential geometry, such as: linearity, derivations, Kähler differentials, differential combinators, tangent vectors, vector bundles, covariant derivatives etc.



The World of Differential Categories - inspired by the graphical representation in "*Tangent Categories from the Coalgebras of Differential Categories*"

[Differential Categories]: They provide the categorical semantics of differential linear logic. They are beyond the scope of this notes. Some references: [BCS06; Blu+20; CLL20].

[Cartesian Differential Categories (CDC)]: They provide the categorical semantics for multivariable differential calculus on Euclidean spaces. The objects of a CDC may be regarded as Euclidean spaces such as $\mathbb{R}^n$ and morphisms, as smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$. A CDC comes equipped with a **differential combinator**, which assigns to each morphism $f\colon \mathbb{R}^n \to \mathbb{R}^m$, a **differential** $\mathsf{D}[f]\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$. $\mathsf{D}[f]$ takes two arguments: a point x in $\mathbb{R}^n$ and a vector u at x, so that $\mathsf{D}[f](x,u)$ is the directional derivative of f at x in the direction of u. Some references: [BCS09].

[Cartesian Differential Restriction Categories (CDRC)]: Like CDCs, CDRCs provide a context for multivariable differential calculus, which, however, is sensitive to partiality. The objects of a CDRC may be regarded as Euclidean spaces such as $\mathbb{R}^n$ and morphisms, smooth partially defined functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$. A CDRC combines two structures: a differential combinator $\mathsf{D}[f]\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, and a restriction structure, which encodes partiality. Some references on restriction categories: [CL02; CL03; CL07]. Some references on CDRCs: [CCG11].

[Generalized Cartesian Differential Categories (GCDC)]: A GCDC is a generalization of a CDC. The differential combinator of a CDC treats the two different inputs of $\mathsf{D}[f]$, the point and the vector, as if they were elements of the same space. This assumption relies on a very peculiar property of Euclidean spaces: in $\mathbb{R}^n$, points and vectors are interchangeable, since at every point corresponds exactly one vector and at every vector, a point. In a GCDC this assumption is dropped: the differential combinator assigns to every morphism $f\colon A \to B$, a morphism $\mathsf{D}[f]\colon A \times \mathsf{L}[A] \to \mathsf{L}[B]$, where $\mathsf{L}[A]$ is a new object equipped with a commutative monoid structure that encodes the space of vectors of A at 0. The objects of a GCDC may be regarded as open subsets $U \subseteq \mathbb{R}^n$ of Euclidean spaces, so that, $\mathsf{L}[U] = \mathbb{R}^n$, and morphisms as smooth functions between those open subsets. Some references: [Cru17].

[Tangent categories]: Tangent categories take the perspective of GCDC of separating between points and vectors at its extreme: not only points and vectors do not live in the same space anymore, but they cannot even be globally distinguished. The objects of a tangent category may be regarded as arbitrary geometric spaces which, however, exhibit a local linear behaviour; the morphisms, as smooth functions between those spaces. References will be found in these notes.

[Tangent Restriction Categories]: Tangent restriction categories combine tangent structures with restriction structures to add a partiality flavour to the geometry. The objects of tangent restriction categories may be regarded as arbitrary geometric spaces which exhibit a local linear behaviour; the morphisms, as smooth partially defined functions between those spaces. Some references: [CC14, Section 6].

3 Tangent categories

When I first studied differential geometry, during my Master's degree in physics at the University of Pavia, it was quite obvious to all the people present that it was going to be a heavy topic. Since I started studying physics, I knew that I was more interested in the mathematical and theoretical aspects, rather than the more experimental aspects of the field. So, I was willing to learn the complicated aspects of differential geometry to better understand the fundamentals of modern physics.

All in all, my experience of differential geometry was positive (I would say, four stars out of five on Google Reviews), however, it was clear to me that the language of this discipline was not easily accessible, since it relies on topology, algebraic topology, differential and integral calculus, analysis, functional analysis, linear algebra, group theory, and many more.

To me, one of the most fascinating aspects of differential geometry was the presence of *structures* that underpin some of the most important constructions and theories of physics. If you study Lagrangian mechanics, you are secretely studying functions on the tangent bundle of a smooth manifold, if you study Hamiltonian mechanics, you are working with Poisson manifolds and symplectic forms. If you are doing anything related to general relativity, you are playing with pseudo-Riemannian geometry, and if you are solving an ordinary differential equation, you are calculating the integral flux of a vector field. If you are studying gauge field theories, principal bundles and principal connections are your friends. I find amusing how fundamental geometry is for physics!

However, some of these important structures used in physics and in other parts of mathematics, are often "obscured" by a long list of assumptions and elements that, when studied the first time, can be a bit cumbersome (this is not mean to be an insult to any differential geometer, here. I'm just reporting my personal experience). For example, it has never been fully clear to me how much important the topological structure of a manifold really is, to develop a theory of geometry. Is the topological structure of a manifold a *necessary* component to do geometry, or is it more a special characteristic of smooth manifolds, that come from a contingent historical development of the theory?

I am remember where the Hausdorff hypothesis is used, or that the paracompactness is assumed to have a partition of unity. However getting these technical assumptions all at once, with the only caveat that "will be useful later" it felt a bit unjustified and overwhelming. Especially because these assumptions are relevant for some constructions, but they aren't for others. It felt a bit unclear to me what aspects of the definition of a manifold were really into play at each step of the theory.

What fundamental structures and axioms are *really* essential for a *theory* of differential geometry?

If physics is the language of nature, differential geometry is the language of physics (at least for classical physics). But what is the language of differential geometry? Is there a simpler, more axiomatic framework to study the fundamental structures of differential geometry?

3.1 A bit of history

Tangent categories were introduced in 1984 by Rosický in a short paper titled *Abstract tangent functors* [Ros84]. Despite being only eleven pages long, in this paper, Rosický already introduced and proved some of the most important definitions and results of the theory. This seminal paper remained hidden for 30 years, until, in 2014, Cockett and Cruttwell rediscovered his work and revitalized the entire project. Nowadays, the tangent category community is very active and is growing in many different directions. The modern definition of a tangent category, on which these notes are based upon and that is more general than Rosický's one, is based on Cockett and Cruttwell's seminal paper [CC14].

3.2 The philosophy of tangent categories

The tangent-first philosophy

Traditional theories of geometry, such as differential or algebraic geometry, follow a *space-first* approach: one starts with model of geometric space, such as a smooth manifold or a scheme, and with that model in mind, one constructs geometric notions, such as the tangent bundle, vector fields, or connections.

The approach of tangent category theory follows a *tangent-first philosophy*. In this perspective, one starts with a collection of abstract objects, interpreted as geometric spaces, equipped with a coherent notion of the tangent bundle. From this primitive notion of the objects and of the tangent bundle, one obtains a geometric interpretation for the objects and construct other geometric notions, such as vector fields, connections etc, from these simple data. In particular, no *a priori* topological or differential structure is required on the objects.

From a logical point of view, tangent categories represent the *theory*, while differential geometry, algebraic geometry and others, are *models* of this theory.

Geometry is contextual

In linear algebra, one defines a *vector* as an element of a vector space. This point of view makes *being a vector* a contextual notion: different vector space-structures on the same set give different notions of being a vector. Category theory is all about context: the context determines the interpretation we give to the objects and the morphisms. The philosophy of tangent categories, applies this concept to geometric spaces: *being a geometric space* is to be an object in a tangent category. In particular, a category might have different tangent structures, and the geometric interpretation depends on this choice. It is worth noting that *every* category is, in fact, a tangent category in a trivial way!

Appreciate what you have, don't desire more

Tangent category theory is not the only categorical approach to differential geometry. Historically, Synthetic Differential Geometry (SDG) has been a leading approach for a while. The main idea of SDG is that the category SMAN of smooth manifolds, is not ideal from a categorical point of view. For instance, pullbacks in SMAN are a mess! Instead, the approach of SDG is to embed SMAN into a larger context, usually a topos, with some extra structure, and develop a theory of geometry in that context. To the contrary, the philosophy of tangent categories is very minimalistic: we work we with what we have. The category SMAN is *itself* a tangent category! We don't make it "nicer" and we only ask for what we need, nothing more, nothing less.

Geometry happens when you separate points from vectors

A CDC (Cartesian Differential Category) is a categorical context for multivariable differential calculus. In particular, a CDC comprises a combinator which assigns to every morphism $f\colon A \to B$ another morphism $\mathsf{D}[f]\colon A\times A \to B$, called the *differential* of f. You can think of $\mathsf{D}[f]$ as the map that takes a point x in A and a vector v in A and sends them to the vector $\mathsf{D}[f](x, v)$ in B which corresponds to the image of the differential of f at x applied to v, that is, $\mathsf{D}[f](x, v) = \mathsf{d}_x f(v)$.

In this context, an object A is regarded in two distinct ways at the same time: as a space of *points* and as a space of *vectors*. This intuition relies on the idea that the objects of a CDC behave as Euclidean spaces, which are precisely those geometric spaces A whose tangent space $\mathrm{T}_x A$ of A at each $x \in A$ is isomorphic to A itself.

In a tangent category, points and vectors belong to entirely distinct spaces! The points live in the base object M, while the vectors of M live in the tangent bundle $\mathrm{T}M$ of M, which is, in fact, interpreted as the space whose points are all tangent vectors of M at every point. In particular, the tangent bundle $\mathrm{T}A$ of a Euclidean space A is isomorphic to $A \times A$.

To make an example, think of the Cartesian plane, which is a Euclidean space. At every point, the tangent space at that point can be identified with the whole space. However, if you think of a sphere, the tangent plane at each point cannot be identified with the whole sphere. The idea of tangent category theory is to keep points and vectors well-separated: *geometry happens when you distinguish between points and vectors.*

Against the regime of fibre bundles

If you have a background of differential geometry (which is not required to understand these notes), you will clearly see that the structure of a tangent category is in correspondence with some concepts of differential geometry: the tangent bundle functor corresponds precisely with the tangent bundle functor in the category SMAN. One important difference is that each tangent bundle $p_M\colon \mathrm{T}M \to M$ in SMAN comes with the structure of a fibre bundle, which is to say that at each point x of M, there is an open neighbourhood U of x whose pre-image $p_M^{-1}(U)$ along p_M is isomorphic to $U \times \mathrm{T}_x M$, where $\mathrm{T}_x M$ is the tangent space of M at x. This is to say that, despite $\mathrm{T}M$ being *globally* very different from $M \times M$, *locally* it can still be regarded as the product of points $y \in U$ and vectors $u \in \mathrm{T}_x M$. Choosing a basis for the vector space $\mathrm{T}_x M$ and one of these open subsets U, is often referred as a choice of *local coordinates* for $\mathrm{T}M$.

In a tangent category this structure is completely missing: we only require the existence of a projection $p_M\colon \mathrm{T}M \to M$ with a bunch of other maps, but there is not mention of a fibre bundle structure on p_M.

We believe that fibre bundles are not required to develop a theory of geometry (at least in the sense of tangent categories). Instead, we think of fibre bundles as consequence of the choice of the geometric spaces of differential geometry. This simplifies the theory greatly!

Real numbers are not real

When I first studied differential geometry in university, I got convinced that to develop a theory of geometry, one cannot dispense from an appropriate notion of *real numbers*. Somehow, real numbers seemed to me to be intrinsically related with the idea of tangent vectors and derivatives. This is why, when I started reading about tangent categories during the COVID 19 pandemic, I was quite skeptical that tangent categories could be a good way to do geometry.

One thing that really bothered me, is that, in a tangent category, we require the tangent bundle $p_M\colon \mathrm{T}M \to M$ to have an additive structure, which defines a zero tangent vector at each point, $z_M\colon M \to \mathrm{T}M$, and it gives a way to sum vectors on the same base point, $s_M\colon \mathrm{T}_2 M \to \mathrm{T}M$. But there is no mention of a scalar action! In particular, the fibres of $p_M\colon \mathrm{T}M \to M$ are required to be commutative monoids, (often we require them also to be Abelian groups), but not vector spaces, like in differential geometry. How can we talk about tangent vectors if $\mathrm{T}_x M$ is not even a vector space?

Tangent categories allow us to define and study an impressive list of geometric concepts. One of these concepts is the notion of a *curve objects*, introduced in [CCL21]. A curve object is a notion of a real-line object! This means that the tangent structure suffices to reconstruct the real numbers. In general however, these objects are not expected in a generic tangent category. When we have a curve object $\mathbb{R}$, the tangent spaces $\mathrm{T}_x M$ acquire an actual scalar action $\mathbb{R} \times \mathrm{T}_x M \to \mathrm{T}_x M$.

A theory is not supposed to capture everything

The main insight of tangent categories is that a primitive notion of the tangent bundle functor (with appropriate structure) sufficies to develop a theory of geometry. However, this doesn't mean that tangent categories can construct *every* geometric notion.

When I studied differential geometry, I recall many times hearing my professors using the word "canonical" or "natural", or also "geometric". Intuitively, a construction is "canonical" or "natural", or "geometric" when it is stable under diffeomorphisms. From a category theory point of view, I now would say that "canonical" translates into "that can be constructed by means of universal properties", and "natural", well, "natural" in the sense of "natural transformation". "Geometric" still means that it is stable under isomorphisms, but it also means, stable under the tangent bundle functor or other operations, like stability under tangent pullbacks.

Tangent category theory is concerned with the study of the "canonical", "natural", and "geometric" constructions of differential geometry, in a setting that does not rely on specific properties or structures of manifolds. In particular, any construction that relies on specific assumptions or properties of spaces, won't be very easy to translate in a general tangent category.

3.3 Building up the definition of a tangent category

The underlying category

The very first step to build the definition of a tangent category is to start with a category. I assume the reader is familiar with this notion, however, maybe, I could spend a couple of words on the interpretation of the underlying category of a tangent category. An object in a tangent category may be regarded as a *generalized geometric space*. This is just an intuitive definition because it will be the tangent structure that will give us the correct interpretation of this concept. Intuitively, one may think of an object M of a tangent category, as a space of points $x, y \in M$. Crucially, such an M should also exhibit a *local linear* behaviour. This means that, at each point, M can be *approximated* by a linear space.

A morphism $f\colon M \to N$ may be interpreted as a map of geometric spaces that can be *approximated* by a linear function.

The tangent bundle functor

One of the philosophical slogans we previously discussed, that drive the research in tangent category sounds like this: "*geometry happens when you separate points from vectors*". Let's first clarify this distinction. Imagine seating at a point x of M and looking around to see whether you might want to go in this or in that direction. x represents a *position*, while the directions are the *tangent vectors* of M at x. If you, like me, have a background in physics, you may want to interpet tangent vectors as the allowed *velocities* of your space, that is, the directions and the speeds at which you can decide to go from a point x.

THE TANGENT BUNDLE FUNCTOR. To keep points (positions) and vectors (velocities) distinct, in a tangent category, we assign to each object M a new object, denoted by $\mathrm{T}M$. Then, M represents the space of points (or positions), while, $\mathrm{T}M$ represents the space of tangent vectors (or velocities) of M. From this perspective, $\mathrm{T}M$ is a given structure. This structure, $\mathrm{T}M$, is informing us of the possible directions and allowed speeds of the space M. I like thinking of $\mathrm{T}M$ as a *traffic code*, a long list of all possible allowed directions of movement (you can go that way, but not that way) together with the allowed speeds you can travel at (yes, you can go above 50km/h).

A good idea is to think of $\mathrm{T}M$ as the space of points (x, v) where x is a point of M and v is a tangent vector of M at x (x is then called the **base point** of v).

The assignment $M \mapsto \mathrm{T}M$ is required to be functorial, that is, to each $f\colon M \to N$ it corresponds a morphism $\mathrm{T}f\colon \mathrm{T}M \to \mathrm{T}N$. $\mathrm{T}f$ sends a point (x, v) in $\mathrm{T}M$ to the point $(f(x), \mathsf{D}[f](x, v))$, where $\mathsf{D}[f](x, v)$ is the *linear approximation* of f at x applied to v. In differential geometry, $\mathsf{D}[f]$ is known as the **differential** of $f\colon M \to N$ at a point. As such, $\mathsf{d}_x f$ sends a tangent vector $v \in \mathrm{T}_x M$ of M at x to a tangent vector $\mathsf{d}_x f(v) \in \mathrm{T}_{f(x)} N$ of N at $f(x)$.

It may be useful to adopt the following notation for $\mathsf{D}[f]$:

$$\mathsf{D}[f](x, v) = \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot v$$

Then, the functoriality of T corresponds to the following two equations:

$$\frac{\mathsf{d}x}{\mathsf{d}x} \cdot v = v \qquad\qquad \frac{\mathsf{d}(g \circ f)}{\mathsf{d}x}(x) \cdot v = \frac{\mathsf{d}g}{\mathsf{d}y}(f(x)) \cdot \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot v$$

The first equation tells us that the linear approximation of the identity is the constant function with value 1, while the second equation is the **chain rule**.

The additive structure

What is left to do is to add the correct structure to $\mathrm{T}M$ to justify us by using the notation (x, v) for $\mathrm{T}M$ and $(f(x), \mathsf{d}f/\mathsf{d}x \cdot v)$ for $\mathrm{T}f(x, v)$.

THE PROJECTION. The first part of this structure is the additive structure of the tangent bundle. The very first ingredient of the additive structure is the **projection**. We require the existence, for every object M, of a map $p_M\colon \mathrm{T}M \to M$. If $\mathrm{T}M$ is the generalized geometric space of tangent vectors of M, then p_M sends each tangent vector (x, v) to its base point $x \in M$. The projection p_M is also required to be a *natural transformation*. This justifies us to write the first component of $\mathrm{T}f(x, v) = (f(x), \mathsf{D}[f](x, v))$ as $f(x)$.

THE ZERO MORPHISM. Next, we assume the existence of a **zero vector field**, that is, a map $z_M\colon M \to \mathrm{T}M$ which is a section of the projection, i.e., $p_M \circ z_M = \mathsf{id}_M$. Thus, z_M is interpreted as the map that sends a point x of M to the zero tangent vector $(x, 0_x)$. As per the projection, z_M is required to be natural. The naturality of z_M may be interpeted in our pointwise notation as the following equation:

$$\frac{\mathsf{d}f}{\mathsf{d}x} \cdot 0_x = 0_{f(x)}$$

THE SUM MORPHISM. Next, we want to be able to sum two vectors together, provided that their base points coincide. To this end, we introduce a **sum morphism** $s_M\colon \mathrm{T}_2M \to \mathrm{T}M$. The domain of s_M, denoted by T_2M, is the geometric space of all pairs of tangent vectors of M having the same base point. Categorically, this can be described using a pullback diagram. In particular, T_2M is the pullback of p_M along itself. Thus, we shall assume that the n-fold pullbacks of the projection

$$\begin{array}{ccc} \mathrm{T}_n M & \xrightarrow{\pi_n} & \mathrm{T}M \\ {\scriptstyle \pi_1}\downarrow & \lrcorner \;\cdot^{\cdot^{\cdot}} & \downarrow{\scriptstyle p_M} \\ \mathrm{T}M & \xrightarrow[p_M]{} & M \end{array}$$

exist in the base category and moreover, that every iterate $\mathrm{T}^m := \mathrm{T} \circ \mathrm{T} \circ \ldots \circ \mathrm{T}$ of the tangent bundle functor would preserve its universality. Concretely, this means that $\mathrm{T}^m \mathrm{T}_n M$ is the n-fold pullback of $\mathrm{T}^m p_M$ along itself. The object $\mathrm{T}_n M$ can allegedly be interpreted as the geometric space of n-tuples of tangent vectors sharing the same base point. We shall use the notation $(x; v_1, \ldots, v_n)$ for the generic point of $\mathrm{T}_n M$.

The sum $s_M : \mathrm{T}_2 M \to \mathrm{T}M$ then, sends $(x; u, v) \in \mathrm{T}_2 M$ to $(x, u + v) \in \mathrm{T}M$. Similarly to the projection and the zero morphism, also the sum is required to be natural. The naturality of s_M translates into the following equation:

$$\frac{\mathsf{d}f}{\mathsf{d}x} \cdot (u + v) = \frac{\mathsf{d}f}{\mathsf{d}x} \cdot u + \frac{\mathsf{d}f}{\mathsf{d}x} \cdot v$$

The sum is also compatible with the projection, that is, the first component of $s_M(x; u, v) = (x, u + v)$ is x, be unital with respect to the zero vector field, that is, $u+0_x = 0_x+u = u$, associative, that is, $(u+v)+w = u+(v+w)$, and commutative, that is, $u + v = v + u$.

We shall give a name for this structure.

Definition 3.1. An **additive bundle** in a category $\mathbb{X}$ consists of a projection, a zero, and a sum morphism

$$q : E \to M \qquad z_q : M \to E \qquad s_q : E_2 \to E$$

respectively, where E_n denotes the n-fold pullback of q along itself, satisfying the following properties:

$$\begin{array}{ccc} E_2 & \xrightarrow{s_q} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow[q]{} & M \end{array} \qquad \begin{array}{ccc} & E_2 & \\ {\scriptstyle \langle \mathsf{id}_E, z_q \circ q \rangle}\nearrow & & \searrow{\scriptstyle s_q} \\ E & = & E \end{array}$$

$$\begin{array}{ccc} & E_2 & \\ {\scriptstyle \langle \pi_2, \pi_1 \rangle}\nearrow & & \searrow{\scriptstyle s_q} \\ E_2 & \xrightarrow[s_q]{} & E \end{array} \qquad \begin{array}{ccc} E_3 & \xrightarrow{\langle \pi_1, s_q \circ \langle \pi_2, \pi_3 \rangle \rangle} & E_2 \\ {\scriptstyle \langle s_q \circ \langle \pi_1, \pi_2 \rangle, \pi_3 \rangle}\downarrow & & \downarrow{\scriptstyle s_q} \\ E_2 & \xrightarrow[s_q]{} & E \end{array}$$

We shall denote an additive bundle by $\mathbf{q} : E \to M$, whose projection is denoted by $q : E \to M$, and the zero and the sum morphisms respectively by $z_q : M \to E$ and $s_q : E_2 \to E$.

Lemma 3.2. *An additive bundle over an object M of $\mathbb{X}$ is equivalent to a commutative monoid in the slice category $\mathbb{X}/M$ with respect to binary products, i.e., pullbacks in $\mathbb{X}$.*

Definition 3.3. An **additive bundle morphism** from an additive bundle $\mathbf{q} : E \to M$ to another additive bundle $\mathbf{q}' : E' \to M'$ consists of a pair of morphisms $f : M \to M'$ and $g : E \to E'$, preserving the projections, the zero, and sum morphisms as follows:

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[f]{} & M' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle z_q}\uparrow & & \uparrow{\scriptstyle z'_q} \\ M & \xrightarrow[f]{} & M' \end{array} \qquad \begin{array}{ccc} E_2 & \xrightarrow{g \times_f g} & E_2 \\ {\scriptstyle s_q}\downarrow & & \downarrow{\scriptstyle s'_q} \\ E & \xrightarrow[g]{} & E' \end{array}$$

By naturality, the triple (p_M, z_M, s_M) defines an additive bundle for each M. We shall call this additive bundle, the **tangent bundle** of M, and we shall denote it by $\mathbf{p}_M\colon \mathrm{T}M \to M$.

NEGATION. In the definition of a tangent category, the tangent bundle $\mathbf{p}_M$ is only required to be an additive bundle. However, we shall often assume the existence of negatives, that is, of a map $n_M\colon \mathrm{T}M \to \mathrm{T}M$, called **negation**. In our pointwise notation, n_M sends (x, v) to $(x, -v)$. As per the projection, the zero, and the sum, n_M is required to be natural. The naturality is equivalent to the following equation:

$$\frac{\mathsf{d}f}{\mathsf{d}x} \cdot (-v) = -\frac{\mathsf{d}f}{\mathsf{d}x} \cdot v$$

The negation n_M makes the commutative monoid $\mathbf{p}_M$ in $\mathbb{X}/M$ into an Abelian group object, by satisfying the following condition. $(-u) + u = 0_x = u + (-u)$.

Definition 3.4. An additive bundle $\mathbf{q}\colon E \to M$ **admits negatives** provided the existence of a map $n_q\colon E \to E$, satisfying the following condition:

$$\begin{array}{ccc} E & \xrightarrow{\langle n_q, \mathsf{id}_E\rangle} & E_2 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s_q} \\ M & \xrightarrow[z_q]{} & E \end{array}$$

Geometric linearity

So far, we have equipped tangent bundle functor with an additive structure. However, this is not enough to allegedly interpret the objects of a tangent category as generalized geometric spaces which exhibit a *local linear* behaviour. But *linear* in what sense?

In linear algebra, a map between vector spaces is linear when it preserves the structure of the vector spaces. Unfortunately, in a tangent category we do not have a real-number object, and the additive structure does not sufficies to make the fibres of the tangent bundle into vector spaces.

Luckily, there is also a more *geometric* definition of linearity. One can say that a function f from $\mathbb{R}$ to $\mathbb{R}$ is linear when it coincides with its first derivative, that is, $f(x) = f'(x)$. This is the perspective of linearity we adopt in tangent categories.

Take into account a Euclidean space, for example think of the Cartesian plane $\mathbb{R}^2$. $\mathbb{R}^2$ carries the structure of an Abelian group. Furthermore, at each point x, the tangent space $\mathrm{T}_x\mathbb{R}^2$ of $\mathbb{R}^2$ at x is canonically isomorphic to the whole space $\mathbb{R}^2$ itself, not only as a geometric space, but also as an Abelian group. Since this is true for each point, we may conclude that the tangent bundle $\mathrm{T}\mathbb{R}^2$ is isomorphic to the Cartesian product $\mathbb{R}^2 \times \mathbb{R}^2$.

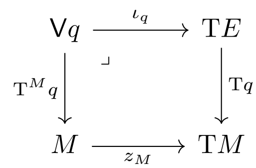
In a tangent category, an object A is *linear* when (1) it carries a commutative monoid structure $(A, 0, +)$ (with respect to the Cartesian product) and (2) $\mathrm{T}A$ is isomorphic to $A \times A$ by an isomorphism $\mathrm{T}A \cong A \times A$ of commutative monoids (notice that $A \times A$ is also a commutative monoid). Later on, we will make this idea more precise with the notion of a *differential object*. It is important to notice that, this notion of linearity is *contextual*, since it is determined by the tangent structure.

In some sense, saying that $\mathrm{T}A \cong A \times A$ is to say that the object A is "equal to its derivative at 0", where the derivative of A, here is the tangent bundle $\mathrm{T}A$.

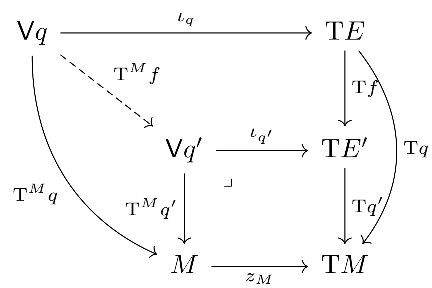
Local linearity

Our goal is to say that every object M in a tangent category exhibit a *local* linear behaviour. So far, we found a way to describe the notion of linearity for objects, however, what does it mean for an object to be *locally* linear? To express this idea of locality, we use the tangent bundle: we say that the tangent bundle $\mathbf{p}_M$ is *itself* a linear object. However, the notion of linearity we are considering, depend on a tangent bundle functor. What tangent bundle functor we should consider to require the tangent bundle $\mathbf{p}_M$ to be linear?

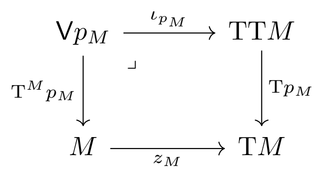
To make the tangent bundle $\mathbf{p}_M$ into an object of another tangent category, we look at the slice category $\mathbb{X}/M$ over M. Let us assume for a moment, that every map admits pullbacks (this is made precise with the notion of tangent display maps [CL25]). Consider an object $q\colon E \to M$ of the slice category $\mathbb{X}/M$. To define a tangent bundle functor $\mathrm{T}^M\colon \mathbb{X}/M \to \mathbb{X}/M$ on the slice category, consider the pullback of $\mathrm{T}q$ along z_M:



Using the universal property of the pullback, we can define an endofunctor T^M on $\mathbb{X}/M$ which sends every q to $\mathrm{T}^M q\colon \mathsf{V}q \to M$ and every morphism $f\colon q \to q'$, that is, $f\colon E \to E'$ such that $q' \circ f = q$, to the unique morphism $\mathrm{T}^M f\colon \mathsf{V}q \to \mathsf{V}q'$ such that:



Now, let us apply this construction to the tangent bundle $\mathbf{p}_M\colon \mathrm{T}M \to M$. $\mathrm{T}^M p_M$ is defined by the following pullback:



If $\mathrm{T}M$ is the space of points (x,v), then $\mathrm{TT}M$ is the space of points $((x,v),(u,\omega))$. Thus, $\mathsf{V}p_M$ becomes the subspace of $\mathrm{TT}M$ of those points $(y;(x,v),(u,\omega))$ satisfying the following equation

$$(y, 0_y) = \frac{\mathsf{d}p_M}{\mathsf{d}(x,v)}(x,v)\cdot(u,\omega) = (x,u)$$

that is, $\mathsf{V}p_M$ is the subspace of $\mathrm{TT}M$ of points $((x,v),(0_x,\omega))$.

The generic point $((x,v),(u,\omega))$ of $\mathrm{TT}M$ is formed by the generic point x of M, a tangent vector v at x and a tangent vector (u,ω) of $\mathrm{T}M$ at (x,v). If a tangent vector v can be regarded as an *infinitesimal path* in a given direction, a tangent vector (u,ω) of $\mathrm{T}M$ can be regarded as an *infinitesimal homotopy*, an infinitesimal path of paths, in M. This can also be regarded as an infinitesimal rectangular tile, whose two sides are determined by the two tangent vectors u and v and whose "filling" is given by something new, the ω.

Let's come back to our goal: force the objects of the underlying category to be *locally* linear. The idea is to require the tangent bundle $\mathbf{p}_M$ to be a *linear* object in the slice category $\mathbb{X}/M$ with respect to the tangent bundle functor T^M. To this end, notice that, by Lemma 3.2, the additive structure of $\mathbf{p}_M$, makes it into a commutative monoid in $\mathbb{X}/M$. Therefore, we only need to require $\mathrm{T}^M\mathbf{p}_M$ to be isomorphic to $\mathbf{p}_M \times_M \mathbf{p}_M$, that is,

$$\mathrm{T}^M\mathbf{p}_M \cong \mathbf{p}_M \times_M \mathbf{p}_M$$

where $\times_M$ is the Cartesian product in $\mathbb{X}/M$ and therefore, the pullback $\mathrm{T}_2M \to M$ of p_M along itself.

THE VERTICAL LIFT. To describe the isomorphism $\mathrm{T}^M\mathbf{p}_M \cong \mathbf{p}_M \times_M \mathbf{p}_M$, we shall introduce a new map, $l_M \colon \mathrm{T}M \to \mathrm{TT}M$, called the **vertical lift**. In our pointwise notation, l_M sends $(x,v) \in \mathrm{T}M$ to $((x,0_x),(0_x,v)) \in \mathrm{TT}M$. As per the natural transformations introduced so far, also the lift is required to be natural. Furthermore, l_M is required to preserves the additive structure of the tangent bundle, that is, $(z_M, l_M) \colon \mathbf{p}_M \to \mathbf{p}_{\mathrm{T}M}$ is required to be an additive bundle morphism.

To encode the *local triviality* of the tangent bundle, we would like to impose the existence of an isomorphism $\mathrm{T}^M\mathbf{p}_M \cong \mathbf{p}_M \times_M \mathbf{p}_M$. From the definition of the tangent bundle functor $\mathrm{T}^M \colon \mathbb{X}/M \to \mathbb{X}/M$ on the slice, we conclude that, requiring the existence of such an isomorphism, is equivalent to force the following diagram to be a pullback

$$\begin{array}{ccc} \mathrm{T}_2M & \xrightarrow{\xi_M} & \mathrm{TT}M \\ {\scriptstyle \pi_1 p_M}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{T}p_M} \\ M & \xrightarrow[z_M]{} & \mathrm{T}M \end{array}$$

where, to define the map ξ_M we need to use the vertical lift as follow:

$$\xi_M \colon \mathrm{T}_2M \xrightarrow{\langle l_M \circ \pi_1, z_{\mathrm{T}M} \circ \pi_2 \rangle} \mathrm{TT}_2M \xrightarrow{\mathrm{T}s_M} \mathrm{TT}M$$

In particular, ξ_M sends $(x; u, v) \in \mathrm{T}_2M$ to $((x,v),(0,u))$.

The symmetry of the Hessian matrix

One of the well-kwown results studied in multivariable differential calculus establishes that the Hessian matrix of a smooth function is always symmetric. Translated in more concrete terms, this means that by taking the directional derivatives of a function $f \colon \mathbb{R}^n \to \mathbb{R}^m$, firstly along a direction u and then along a direction v, that is, by computing $\partial_v\partial_u f$, is equivalent to first compute the directional derivative of f along v and then along u, that is, $\partial_v\partial_u f$ is equal to $\partial_u\partial_v f$.

In differential geometry, one can see tangent vectors as infinitesimal smooth paths from a given point towards a specified direction and with a given initial velocity. Another equivalent way, is to think of a tangent vector at a point x of a geometric space M as a derivation operator, that is, a linear operator

$$\mathsf{D}_v \colon \mathscr{C}^\infty(M) \to \mathbb{R}$$

defined on the algebra of real-valued smooth functions $\mathscr{C}^\infty(M)$ into $\mathbb{R}$, satisfying the Leibniz rule:

$$\mathsf{D}_v(f \cdot g) = f(x)\mathsf{D}_v(g) + g(x) \cdot \mathsf{D}_v(f)$$

in particular, $\mathsf{D}_v(f)$ is the directional derivative of f at x along the direction v. With this idea, one can think of the elements of $\mathrm{TT}M$, as second directional derivatives $\mathsf{D}_{u,v}f$.

THE CANONICAL FLIP. To encode the symmetry $\partial_u\partial_v = \partial_v\partial_u$ we introduce an isomorphism $c_M \colon \mathrm{TT}M \to \mathrm{TT}M$, called the **canonical flip**. In our pointwise notation, c_M sends the generic point $((x,u),(v,\omega))$ of $\mathrm{TT}M$ to $((x,v),(u,\omega))$, by flipping the internal components. The flip is also required to be natural, and this condition translates into precisely the desired symmetry $\partial_u\partial_v f = \partial_v\partial_u f$.

Since the tangent bundle functor preserves the n-fold pullbaks of the projection along itself, T sends the additive bundle $\mathbf{p}_M \colon \mathrm{T}M \to M$ to a new additive bundle $\mathrm{T}\mathbf{p}_M \colon \mathrm{TT}M \to \mathrm{T}M$. Thus, c_M defines an additive bundle isomorphism $(\mathsf{id}_{\mathrm{T}M}, c_M) \colon \mathbf{p}_{\mathrm{T}M} \to \mathrm{T}\mathbf{p}_M$.

The full definition

We may now recollect everything we have said so far into a full definition.

Definition 3.5 ([Ros84],[CC14, Definition 2.3])**.** A **tangent structure** on a category $\mathbb{X}$ consists of an endofunctor $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$, called the **tangent bundle functor** together with a collection of **structural natural transformations**

$$p_M\colon \mathrm{T}M \to M \qquad z_M\colon M \to \mathrm{T}M \qquad s_M\colon \mathrm{T}_2M \to \mathrm{T}M$$
$$l_M\colon \mathrm{T}M \to \mathrm{T}\mathrm{T}M \qquad\qquad c_M\colon \mathrm{T}\mathrm{T}M \to \mathrm{T}\mathrm{T}M$$

natural in M, respectively called, the **projection**, the **zero morphism**, the **sum morphism**, the **vertical lift**, and the **canonical flip**, where T_nM denotes the n-fold pullback of p_M along itself, subject to the following conditions:

[TAN.1]: The n-fold pullback

$$\begin{array}{ccc} \mathrm{T}_nM & \xrightarrow{\pi_n} & \mathrm{T}M \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner\ \ddots & \big\downarrow{\scriptstyle p_M} \\ \mathrm{T}M & \xrightarrow[p_M]{} & M \end{array}$$

of p_M along itself exists for every $n \geq 0$ and is an n-fold **tangent pullback**, that is, is preserved by all iterates T^m of the tangent bundle functor;

[TAN.2]: For each object $M \in \mathbb{X}$, $\mathbf{p}_M := (p_M, z_M, s_M)$ is an addtive bundle, called the **tangent bundle** of M;

[TAN.3]: The vertical lift is additive, that is, $(z_M, l_M)\colon \mathbf{p}_M \to \mathrm{T}\mathbf{p}_M$ is an additive bundle morphism;

[TAN.4]: The canonical flip is additive, that is, $(\mathsf{id}_{\mathrm{T}M}, c_M)\colon \mathbf{p}_{\mathrm{T}M} \to \mathrm{T}\mathbf{p}_M$ is an additive bundle morphism;

[TAN.5]: The vertical lift is coassocative and compatible with the canonical flip:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{l_M} & \mathrm{T}^2M \\ {\scriptstyle l_M}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{T}l_M} \\ \mathrm{T}^2M & \xrightarrow[l_{\mathrm{T}M}]{} & \mathrm{T}^3M \end{array} \qquad \begin{array}{ccc} \mathrm{T}M & \xrightarrow{l_M} & \mathrm{T}^2M \\ & {\scriptstyle l_M}\searrow & \big\downarrow{\scriptstyle c_M} \\ & & \mathrm{T}^2M \end{array} \qquad \begin{array}{ccccc} \mathrm{T}^2M & \xrightarrow{l_{\mathrm{T}M}} & \mathrm{T}^3M & \xrightarrow{\mathrm{T}c_M} & \mathrm{T}^3M \\ {\scriptstyle c_M}\big\downarrow & & & & \big\downarrow{\scriptstyle c_{\mathrm{T}M}} \\ \mathrm{T}^2M & & \xrightarrow[\mathrm{T}l_M]{} & & \mathrm{T}^3M \end{array}$$

[TAN.6]: The canonical flip is a symmetric braiding:

$$\begin{array}{ccc} \mathrm{T}^2M & \xrightarrow{c_M} & \mathrm{T}^2M \\ & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle c_M} \\ & & \mathrm{T}^2M \end{array} \qquad \begin{array}{ccccc} \mathrm{T}^3M & \xrightarrow{\mathrm{T}c_M} & \mathrm{T}^3M & \xrightarrow{c_{\mathrm{T}M}} & \mathrm{T}^3M \\ {\scriptstyle c_{\mathrm{T}M}}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{T}c_M} \\ \mathrm{T}^3M & \xrightarrow[\mathrm{T}c_M]{} & \mathrm{T}^3M & \xrightarrow[c_{\mathrm{T}M}]{} & \mathrm{T}^3M \end{array}$$

[TAN.7]: The tangent bundle is **locally linear**, that is, the following is a pullback diagram

$$\begin{array}{ccc} \mathrm{T}_2M & \xrightarrow{\xi_M} & \mathrm{T}\mathrm{T}M \\ {\scriptstyle \pi_1 p_M}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{T}p_M} \\ M & \xrightarrow[z_M]{} & \mathrm{T}M \end{array}$$

where ξ_M is defined as follows:

$$\xi_M\colon \mathrm{T}_2M \xrightarrow{\langle l_M \circ \pi_1, z_{\mathrm{T}M} \circ \pi_2 \rangle} \mathrm{T}\mathrm{T}_2M \xrightarrow{\mathrm{T}s_M} \mathrm{T}\mathrm{T}M$$

A **tangent category** consists of a category $\mathbb{X}$ equipped with a tangent structure $\mathbb{T} := (\mathrm{T}, p, z, s, l, c)$. Furthermore, a tangent structure **has negatives** when it is equipped with an extra natural transformation

$$n_M \colon \mathrm{T}M \to \mathrm{T}M$$

natural in M, called **negation**, subject to the following condition:

[TAN$^-$]: n_M makes each tangent bundle $\mathbf{p}_M$ into an additive bundle with negatives, by satisfying the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\langle n_M, \mathsf{id}_{\mathrm{T}M}\rangle} & \mathrm{T}_2M \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle s_M} \\ M & \xrightarrow[z_M]{} & \mathrm{T}M \end{array}$$

3.4 Some examples of tangent categories

Every category is a tangent category

It turns out that every category comes equipped with a tangent structure, called the **trivial tangent structure**, where the tangent bundle functor and the structural natural transformations are the identities. This example teaches us something important: following the philosophy of tangent categories, *geometry is contextual*, not intrinsic to the objects. In particular, the trivial tangent structure corresponds to the trivial geometric theory, in which the only available direction is the zero direction. This is the theory of the "points that don't move". Another way to put it, is the trivial geometry, the spaces can be seen as discrete spaces, where every point is isolated to anyone else and no movement from one point to another is allowed.

The tangent category of differential geometry

The category SMAN of finitely-dimensional smooth manifolds represents the archetypal example of a tangent category, since the definition of the tangent bundle functor and associated structural natural transformations are directly abstracted from SMAN. In particular, the tangent bundle $\mathrm{T}M$ of a smooth manifold M is the manifold of all tangent vectors of M, where a tangent vector at $x \in M$ is defined as an equivalence class of smooth paths $\gamma \colon \mathbb{R} \to M$ satisfying the condition $\gamma(0) = x$ and where $\gamma \sim \theta$ if and only if $\gamma'(0) = \theta'(0)$ (where γ' and θ' are the first derivatives of γ and θ). Equivalently, tangent vectors can also be defined as $\mathbb{R}$-linear operators $\mathsf{D} \colon \mathscr{C}^\infty(M) \to \mathbb{R}$ satisfying $\mathsf{D}(f \cdot g) = f(x) \cdot \mathsf{D}(g) + g(x) \cdot \mathsf{D}(f)$. The additive structure is given by the additive structure of the tangent vector spaces T_xM. The vertical lift sends a tangent vector (x, v) to the tangent vector corresponding to the derivation operator

$$\hat{\mathsf{D}}_h[g] := \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} g(x, tv)$$

for each $g \in \mathscr{C}^\infty(\mathrm{T}M)$.

The tangent structure generated by biproducts

Consider now a category $\mathbb{X}$ equipped with biproducts, that is, $\mathbb{X}$ admits finite products, finite coproducts, and (1) the terminal object is initial (there is a zero object), and (2) binary products are isomorphic to the binary coproducts along the canonical map $[\langle \mathsf{id}_A, 0\rangle, \langle 0, \mathsf{id}_B\rangle] \colon A + B \to A \times B$. We shall denote the biproducts by $\oplus$.

The tangent bundle functor $\mathrm{T}^{\oplus}$ sends A to $A \oplus A$ and $f\colon A \to B$ to $f \oplus f$. The additive structure is defined as follows

$$p_A^{\oplus}\colon A \oplus A \xrightarrow{\pi_1} A \qquad z_A^{\oplus}\colon A \xrightarrow{\iota_1} A \oplus A \qquad s_A^{\oplus}\colon A \oplus A \oplus A \xrightarrow{\mathsf{id}_A \oplus +_A} A \oplus A$$

where $+_A := [\mathsf{id}_A, \mathsf{id}_A]\colon A \oplus A \to A$, and the vertical lift and the canonical flip are defined by

$$l_A^{\oplus}\colon A \oplus A \xrightarrow{\iota_1 \oplus \iota_2} (A \oplus A) \oplus (A \oplus A) \qquad c_A^{\oplus}\colon A \oplus (A \oplus A) \oplus A \xrightarrow{\mathsf{id}_A \oplus \tau_{A,A} \oplus \mathsf{id}_A} A \oplus (A \oplus A) \oplus A$$

where $\tau_{A,B}\colon A \oplus B \to B \oplus A$ is the canonical symmetry. The existence of biproducts makes $\mathbb{X}$ enriched over commutative monoids. When $\mathbb{X}$ is also enriched over Abelian groups, then the tangent structure $\mathbb{T}^{\oplus}$ admits negatives:

$$n_A^{\oplus}\colon A \oplus A \xrightarrow{\mathsf{id}_A \oplus -_A} A$$

where $-_A$ denotes $-\mathsf{id}_A$ in the Hom-Abelian group $\mathbb{X}(A, A)$.

The tangent structure of commutative and unital algebras

Consider the category CALG_R of commutative and unital algebras over a commutative and unital rig (a.k.a. semiring) R and denote by $R[\varepsilon]$ the ring of dual numbers, that is, the R-algebra $R[x]/(x^2)$, so ε denotes the left coset $\varepsilon = x + R[x]$, $\varepsilon^2 = 0$, and every element of $R[\varepsilon]$ is of the form $a + u\varepsilon$, for $a, u \in R$. Notice that $R[\varepsilon]$ comes with the following maps:

$$\begin{aligned}
&p^{\varepsilon}\colon R[\varepsilon] \to R && p^{\varepsilon}(a + u\varepsilon) = a\\
&z^{\varepsilon}\colon R \to R[\varepsilon] && z^{\varepsilon}(a) = a = a + 0\varepsilon\\
&s^{\varepsilon}\colon R[\varepsilon_1, \varepsilon_2] \to R[\varepsilon] && s^{\varepsilon}(a + u\varepsilon_1 + v\varepsilon_2) = a + (u+v)\varepsilon\\
&l^{\varepsilon}\colon R[\varepsilon] \to R[\varepsilon] \otimes R[\varepsilon] && l^{\varepsilon}(a + u\varepsilon) = a + u(\varepsilon \otimes \varepsilon)\\
&c^{\varepsilon}\colon R[\varepsilon] \otimes R[\varepsilon] \to R[\varepsilon] \otimes R[\varepsilon] && c^{\varepsilon}(a + u(\varepsilon \otimes 1) + v(1 \otimes \varepsilon) + \omega(\varepsilon \otimes \varepsilon)) = a + v(\varepsilon \otimes 1) + u(1 \otimes \varepsilon) + \omega(\varepsilon \otimes \varepsilon)
\end{aligned}$$

where $R[\varepsilon_1, \varepsilon_2] = R[x, y]/(x^2, y^2, xy)$ is the pullback of p^{ε} along itself. From this, it follows that the functor T^{ε} which sends $A \in \mathsf{CALG}$ to $\mathrm{T}^{\varepsilon}A := R[\varepsilon] \otimes A \cong A[\varepsilon]$ together with the corresponding natural transformations obtained by tensoring the morphisms defined previously by each A, defines a tangent structure denoted by $\mathbb{T}^{\varepsilon}$.

When R is a ring, not just a rig, then $R[\varepsilon]$ comes also equipped with a negation morphism

$$n^{\varepsilon}\colon R[\varepsilon] \to R[\varepsilon] \qquad n^{\varepsilon}(a + u\varepsilon) = a - u\varepsilon$$

which induces negatives on the tangent structure $\mathbb{T}^{\varepsilon}$.

The tangent categories of algebraic geometry

One of the currently active projects in the community of tangent categories is the idea of using tangent categories to study algebraic geometry. Some of the most active people in this project are Geoffrey Cruttwell, JS Lemay, and Geoff Vooys [CL23; LV25]. In this section, we only focus on affine schemes, while we leave it to the reader to read the extended version on schemes in the literature.

This section in these notes is inspired by Geoffrey Cruttwell's beautiful notes on this topic that you can download from here: https://www.reluctantm.com/gcruttw/publications/alg_geo_notes.pdf.

The first main idea of this approach is to entirely avoid to talk about affine schemes and instead focusing directly on the opposite category of commutative and unital algebras. The equivalence between the category of affine schemes over a ring R and $\mathsf{CALG}_R^{\mathsf{op}}$ is a well-known result of algebraic geometry.

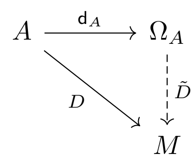
The underlying category of the *tangent category of affine schemes* is $\mathsf{CALG}_R^{\mathsf{op}}$. The tangent bundle functor T^Ω sends an algebra A to the symmetric A-algebra of the module of Kähler differentials of A, that is:

$$\mathrm{T}^\Omega A := \mathsf{Sym}_A \Omega_A = \bigoplus_{n\geq 0} \Omega_A^{\otimes_A n} = A \oplus \Omega_A \oplus (\Omega_A \otimes_A \Omega_A) \oplus (\Omega_A \otimes_A \Omega_A \otimes_A \Omega_A) \oplus \dots$$

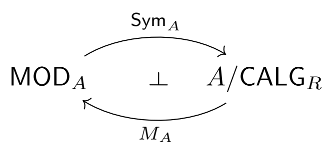
Let us unpack this definition. The **module of Kähler differentials** of an algebra A is the A-module Ω_A equipped with the universal derivation $\mathsf{d}_A \colon A \to \Omega_A$. Concretely, this means that, given a derivation over an A-module M, that is, an R-linear map $D \colon A \to M$ satisfying the Leibniz equation

$$D(a \cdot b) = a \cdot D(b) + b \cdot D(a)$$

there exists a unique $\tilde{D} \colon \Omega_A \to M$ of A-modules (so, $\tilde{D}$ is A-linear) such that



It turns out that Ω_A can also be expressed as the kernel of the multiplication map $\cdot \colon A \otimes A \to A$ of A. Sym_A is the functor that sends an A-module M to the universal algebra under A, that is, an algebra morphism $A \to \mathsf{Sym}_A M$. In particular, Sym_A can be understood as the left adjoint of the functor $M_A \colon A/\mathsf{CALG}_R \to \mathsf{MOD}_A$ which sends a morphism $f \colon A \to B$ to the A-module B induced by f:



More concretely, $\mathrm{T}^\Omega A$ is the R-algebra freely generated by the elements $a, b, \dots$ of A together with symbols $\mathsf{d}a$, for each $a \in A$, subject to the following relations:

$$a \cdot_{\mathrm{T}^\Omega A} b = a \cdot_A b \qquad \mathsf{d}(ra + sb) = r\mathsf{d}a + s\mathsf{d}b$$
$$\mathsf{d}1 = 0 \qquad \mathsf{d}(a \cdot b) = a.\mathsf{d}b + b \cdot \mathsf{d}a$$

The structural natural transformations (written as algebra morphisms, remember that we are in the opposite category) are defined as follows

$$p_A^\Omega \colon A \to \mathrm{T}^\Omega A \qquad p_A^\Omega(a) := a$$
$$z_A^\Omega \colon \mathrm{T}^\Omega A \to A \qquad z_A^\Omega(a) = a, \quad z^\Omega(\mathsf{d}a) = 0$$
$$s_A^\Omega \colon \mathrm{T}^\Omega A \to \mathrm{T}^\Omega A \otimes_A \mathrm{T}^\Omega A \qquad s_A^\Omega(a) = a, \quad s_A^\Omega(\mathsf{d}a) = \mathsf{d}a \otimes 1 + 1 \otimes \mathsf{d}a$$

where the pullback of the projection along itself in $\mathsf{CALG}_R^{\mathsf{op}}$ is a pushout in CALG_R and it coincides with the tensor product $\mathrm{T}^\Omega A \otimes_A \mathrm{T}^\Omega A$ over A. To understand the vertical lift and the canonical flip, first, notice that $\mathrm{T}^\Omega \mathrm{T}^\Omega A$ is generated by each a, $\mathsf{d}a$, $\mathsf{d}'a$, and by $\mathsf{d}'\mathsf{d}a$ with appropriate relations. So, we have:

$$l_A^\Omega \colon \mathrm{T}^\Omega \mathrm{T}^\Omega A \to \mathrm{T}^\Omega A \qquad l_A^\Omega(a) = a, \quad l_A^\Omega(\mathsf{d}a) = l_A^\Omega(\mathsf{d}'a) = 0, \quad l_A^\Omega(\mathsf{d}'\mathsf{d}a) = \mathsf{d}a$$
$$c_A^\Omega \colon \mathrm{T}^\Omega \mathrm{T}^\Omega A \to \mathrm{T}^\Omega \mathrm{T}^\Omega A \qquad c_A^\Omega(a) = a, \quad c_A^\Omega(\mathsf{d}a) = \mathsf{d}'a, \quad c_A^\Omega(\mathsf{d}'a) = \mathsf{d}a, \quad c_A^\Omega(\mathsf{d}'\mathsf{d}a) = \mathsf{d}'\mathsf{d}a)$$

When the base rig R has negatives, then we also define a negation:

$$l_A^\Omega \colon \mathrm{T}^\Omega A \to \mathrm{T}^\Omega A \qquad n_A^\Omega(a) = a, n_A^\Omega(\mathsf{d}a) = -\mathsf{d}a$$

We shall denote this tangent structure by $\mathbb{T}^\Omega$.

To have a better understanding of this tangent structure, we shall turn to some examples.

Example 3.6. Let us start with the algebra R. Since $\mathsf{d}1 = 0$, the module of Kähler differentials of R is 0, thus, $\mathrm{T}^{\Omega}R = R$. Geometrically, this can be translated by saying that the tangent bundle of a point is just the point itself.

Example 3.7. Now consider with the algebra $R[x]$. Then, $\mathrm{T}^{\Omega}R[x]$ is generated by all $p(x) \in R[x]$ and by the $\mathsf{d}p(x)$, for each $p(x)$. Let us compute what is $\mathsf{d}p(x)$, for a given polynomial $p(x) = \sum_k a_k x^k$:

$$\mathsf{d}p(x) = \mathsf{d}\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} a_k \mathsf{d}(x^k) = a_0\mathsf{d}(1) + \sum_{k=1}^{n} a_k \mathsf{d}(x^k) = \sum_{k=1}^{n} a_k k x^{k-1}\mathsf{d}x = \sum_{k=0}^{n-1} a_{k+1}(k+1)x^k \mathsf{d}x$$

Therefore, $\mathrm{T}^{\Omega}R[x] = R[x] \otimes R[x]\mathsf{d}x = R[x, \mathsf{d}x]$.

Example 3.8. We can generalize Example 3.7 and obtain:

$$\mathrm{T}^{\Omega}R[x_1, \ldots, x_n] = R[x_1, \ldots, x_n, \mathsf{d}x_1, \ldots, \mathsf{d}x_n]$$

Example 3.9. One can show that the tangent bundle functor T^{Ω} is a right adjoint in $\mathsf{CALG}_R^{\mathsf{op}}$, thus, a left adjoint in CALG_R. Therefore, T^{Ω} preserves colimits, and in particular, coequalizers. This abstract reasoning tells us that if A is obtained as the quotient of a free algebra F over an ideal I, then $\mathrm{T}^{\Omega}A = \mathrm{T}^{\Omega}(F/I) = (\mathrm{T}^{\Omega}F)/(\mathrm{T}^{\Omega}I)$. It will be clear in a second what we mean with that.

Consider the algebra $C = R[x, y]/(x^2 + y^2 - 1)$. Thus:

$$\mathrm{T}^{\Omega}C = (\mathrm{T}^{\Omega}R[x, y])/(x^2 + y^2 - 1, \mathsf{d}(x^2 + y^2 - 1)) = R[x, y, \mathsf{d}x, \mathsf{d}y]/(x^2 + y^2 - 1, 2x\mathsf{d}x + 2y\mathsf{d}y)$$

When we are not in characteristic 2, we can also write it as $R[x, y, \mathsf{d}x, \mathsf{d}y]/(x^2 + y^2 - 1, x\mathsf{d}x + y\mathsf{d}y)$. Notice that, to compute the relations of $\mathrm{T}^{\Omega}(F/I)$, we differentiated the relations expressed in the ideal I.

To understand if this construction is in any ways meaningful, we can start by calculate the tangent space of an affine scheme at a given point. For starters, we need to introduce the notion of a tangent space at a given point.

Definition 3.10. In a tangent category $(\mathbb{X}, \mathbb{T})$, the **tangent space** of an object M at a given point $x \colon * \to M$ (where $*$ denotes the terminal object) is the tangent pullback of the tangent bundle $\mathbf{p}_M$ of M along x, that is:

$$\begin{array}{ccc} \mathrm{T}_x M & \xrightarrow{\iota_x} & \mathrm{T}M \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_M} \\ * & \xrightarrow[x]{} & M \end{array}$$

We shall assume that all tangent pullbacks of the projection along arbitrary maps may exist in a tangent category. In the case of $\mathsf{CALG}_R^{\mathsf{op}}$, the category is complete and the tangent bundle functor is a right adjoint. A point in $\mathsf{CALG}_R^{\mathsf{op}}$, a.k.a. global element, of A corresponds to a map $A \to R$ of algebras.

Example 3.11. Consider again the algebra $C = R[x, y]/(x^2 + y^2 - 1)$. A point of C consists of a map $f \colon C \to R$, which is fully specified by two numbers $a := f(x)$ and $b := f(y)$. Let us denote f by (a, b) Furthermore, to be a morphism of algebras, a and b must satisfy the equation $a^2 + b^2 - 1$, which means that a point of C corresponds precisely to a point on the circle, in the usual sense. Now, let us compute the tangent space of C at the point (a, b). The pullback in CALG_R becomes the following pushout:

$$\begin{array}{ccc} C & \xrightarrow{(a,b)} & R \\ {\scriptstyle p_C^{\Omega}}\downarrow & \ulcorner & \downarrow \\ \mathrm{T}^{\Omega}C & \longrightarrow & \mathrm{T}^{\Omega}C \otimes_C R \end{array}$$

Concretely, tensoring $\mathrm{T}^{\Omega}C$ by R along C translates into assigning the values a and b to the variables x and y, respectively:

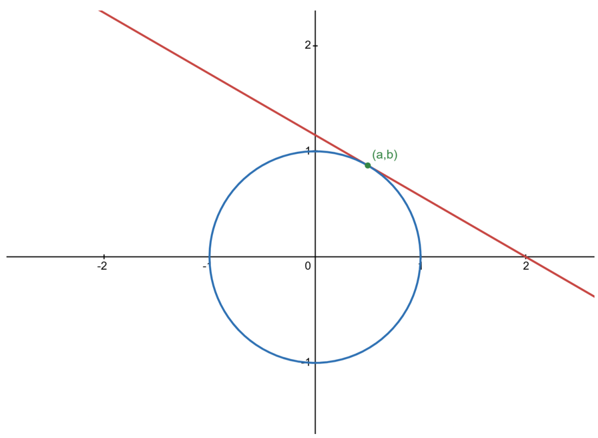
$$\mathrm{T}^{\Omega}_{(a,b)}C = \frac{R[x, y, \mathsf{d}x, \mathsf{d}y]}{(x^2 + y^2 - 1, 2x\mathsf{d}x + 2y\mathsf{d}y, x - a, y - b)} = \frac{R[\mathsf{d}x, \mathsf{d}y]}{(2a\mathsf{d}x + 2b\mathsf{d}y)}$$

When we are not in characteristic 2, $R[\mathsf{d}x, \mathsf{d}y]/(2a\mathsf{d}x + 2b\mathsf{d}y)$ is isomorphic to $R[z]$ where the isomorphism is constructed as follows. If $a \neq 0$, then it sends $\mathsf{d}y$ to z and $\mathsf{d}x$ to $-(b/a)z$; if $b \neq 0$, then it sends $\mathsf{d}x$ to z and $\mathsf{d}y$ to $-(a/b)z$; a and b cannot vanish at the same time, since $a^2 + b^2 = 1$.

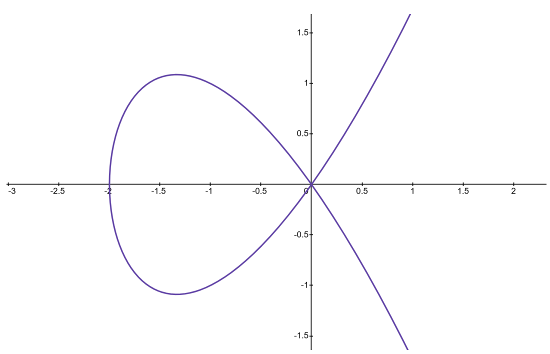
Let for example $R = \mathbb{R}$ be the ring of real numbers and consider the point $a = 1/2$ and $b = \sqrt{3}/2$ of the circle. Thus:

$$\mathrm{T}^{\Omega}_{(1/2,\sqrt{3}/2)}C = \frac{R[\mathsf{d}x, \mathsf{d}y]}{\left(\frac{1}{2}\mathsf{d}x + \frac{\sqrt{3}}{2}\mathsf{d}y\right)}$$

Notice that the equation $\frac{1}{2}(x - \frac{1}{2}) + \frac{\sqrt{3}}{2}(y - \frac{\sqrt{3}}{2}) = 0$ is the equation of the tangent line of the circle at $(1/2, \sqrt{3}/2)$, where we simply replaced $\mathsf{d}x$ with the displacement $(x - 1/2)$ and $\mathsf{d}y$ with $(y - \sqrt{3}/2)$:



Example 3.12. Now, we consider another algebra associated to a polynomial curve: $B := R[x, y]/(y^2 - x^3 - 2x^2)$. The graph of the curve of equation $y^2 - x^3 - 2x^2 = 0$ is plot here:



Something interesting happens at point $(0, 0)$: while at every other point the tangent space is 1-dimensional, at $(0, 0)$ the tangent space seems to be 2-dimensional. Let us compute the tangent bundle of B:

$$\mathrm{T}^{\Omega}B = \frac{R[x, y, \mathsf{d}x, \mathsf{d}y]}{(y^2 - x^3 - 2x^2, \mathsf{d}(y^2 - x^3 - 2x^2))} = \frac{R[x, y, \mathsf{d}x, \mathsf{d}y]}{(y^2 - x^3 - 2x^2, 2y\mathsf{d}y - 3x^2\mathsf{d}x - 4x\mathsf{d}x)}$$

Consider a point $(a, b)\colon B \to R$, where $a, b \in R$ satisfy $b^2 - a^3 - 2a^2 = 0$. Thus, the tangent space of B at (a, b) is given by:

$$\mathrm{T}^{\Omega}_{(a,b)}B \frac{R[x, y, \mathsf{d}x, \mathsf{d}y]}{(y^2 - x^3 - 2x^2, 2y\mathsf{d}y - 3x^2\mathsf{d}x - 4x\mathsf{d}x, x - a, y - b)} = \frac{R[\mathsf{d}x, \mathsf{d}y]}{(2b\mathsf{d}y - 3a^2\mathsf{d}x - 4a\mathsf{d}x)}$$

When neither a and b vanish, the equation $2b\mathsf{d}y - 3a^2\mathsf{d}x - 4a\mathsf{d}x = 0$ forces a non-trivial relation between the variables $\mathsf{d}x$ and $\mathsf{d}y$, rendering $\mathrm{T}^{\Omega}_{(a,b)}B$ one-dimensional, as expected. However, when both a and b vanish, that is, $(a, b) = (0, 0)$, the equation $2b\mathsf{d}y - 3a^2\mathsf{d}x - 4a\mathsf{d}x = 0$ is solved for every value of $\mathsf{d}x$ and $\mathsf{d}y$, namely, $\mathrm{T}^{\Omega}_{(0,0)}B = R[\mathsf{d}x, \mathsf{d}y]$ is 2-dimensional, as expected!

4 Geometric constructions with tangent categories

One of the active programs in tangent category theory aims to internalize as much as geometric constructions from differential geoemtry as possible in tangent categories. We already have a long list and notions that have been successfully internalized. This list includes: vector fields, Euclidean spaces, vector bundles, Koszul connections and related notions such as curvature, torsion, and covariant derivative, Lie algebroids, submersions, immersions, étale maps, ordinary differential equations, and dynamical systems.

This offers a rich language of geometric "building blocks" for a full theory of geometry. As part of this program, one also would like to compute these constructions in other settings, such as in algebraic geometry. In this section, we explore three important constructions: vector fields, differential objects, and differential bundles, which respectively generalize ordinary vector fields, Euclidean spaces, and vector bundles.

4.1 Vector fields

Intuitively, a vector field on a geometric space M, consists of an assignment, for each point x, of a tangent vector at x. One may regard a vector field as a way to assign directions and speed limits at every point of the space. Vector fields were first introduced in tangent categories by Rosický [Ros84].

Definition 4.1 ([Ros84])**.** A **vector field** on an object M in a tangent category consists of a morphism $v\colon M \to \mathrm{T}M$ which is a section of the projection, that is, $p_M \circ v = \mathsf{id}_M$.

Every object of a tangent category has a canonical vector field, the zero vector field $z_M\colon M \to \mathrm{T}M$, since $p_M \circ z_M = \mathsf{id}_M$. Furthermore, given two vector fields u, v on M, we can define a new vector field, denoted by $u + v$, as follows:

$$u + v\colon M \xrightarrow{\langle u,v\rangle} \mathrm{T}_2M \xrightarrow{s_M} \mathrm{T}M$$

In general, for two maps $f, g\colon M \to \mathrm{T}N$, satisfying $p_N \circ f = \mathbf{p}_N \circ g$, we can define $f + g\colon N \to \mathrm{T}N$ as follows:

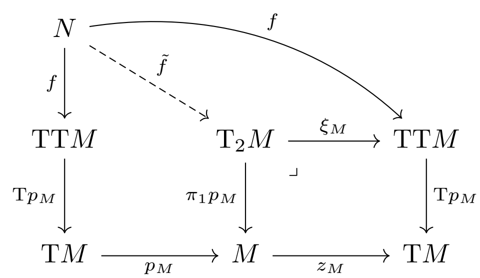
$$f + g\colon M \xrightarrow{\langle f,g\rangle} \mathrm{T}_2N \xrightarrow{s_N} \mathrm{T}N$$

When the tangent category has negatives, for each map $f\colon M \to \mathrm{T}N$, we can also define

$$-f\colon M \xrightarrow{f} \mathrm{T}N \xrightarrow{n_N} \mathrm{T}N$$

Lemma 4.2. *The set* $\mathsf{VF}(\mathbb{X}, \mathbb{T}; M)$ *of vector fields over* M *in a tangent category* $(\mathbb{X}, \mathbb{T})$ *comes equipped with a commutative monoid structure whose zero is* $0 = z_M$ *and there the sum is defined by* $u + v$*. Moreover, when* $(\mathbb{X}, \mathbb{T})$ *admits negatives,* $\mathsf{VF}(\mathbb{X}, \mathbb{T}; M)$ *is an Abelian group.*

In differential geometry, the Abelian group $\mathsf{VF}(\mathbb{X},\mathbb{T};M)$ comes also equipped with a Lie bracket, forming a Lie algebra. This is an important result of differential geometry. It turns out that this structure is also present in any tangent category with negatives! Let us see how to define this notion. For starters, consider a morphism $f\colon N\to \mathrm{TT}M$ satisfying the following condition. $\mathrm{T}p_M\circ f = z_M\circ p_M\circ \mathrm{T}p_M\circ f$. Under this condition, we can use the universal property of the vertical lift and construct a unique morphism $\tilde{f}\colon N\to \mathrm{T}_2M$ that makes the following diagram commutative:



We denote by $\{f\}\colon N\to \mathrm{T}M$, the morphism:

$$\{f\}\colon N\xrightarrow{\tilde{f}}\mathrm{T}_2M\xrightarrow{\pi_1}\mathrm{T}M$$

Lemma 4.3. *Consider two vector fields u and v on an object M in a tangent category with negatives. Then, the morphism $[\![u,v]\!] := (\mathrm{T}v\circ u) - (c_M\circ \mathrm{T}u\circ v)$ satisfies the following equation:*

$$\mathrm{T}p_M\circ [\![u,v]\!] = z_M\circ p_M\circ \mathrm{T}p_M\circ [\![u,v]\!]$$

Proof. This is a simple computation:

$$\begin{aligned}
\mathrm{T}p_M\circ [\![u,v]\!] &= \mathrm{T}p_M\circ((\mathrm{T}v\circ u) - (c_M\circ \mathrm{T}u\circ v)) \\
&= (\mathrm{T}p_M\circ \mathrm{T}v\circ u) - (\mathrm{T}p_M\circ c_M\circ \mathrm{T}u\circ v) && (\text{Naturality of } n_M)\\
&= u - (p_{\mathrm{T}M}\circ \mathrm{T}u\circ v) && (p_M\circ v_M=\mathsf{id}_M, \mathrm{T}p_M\circ c_M = p_{\mathrm{T}M})\\
&= u - (u\circ p_M\circ v) && (\text{Naturality of } p_M)\\
&= u-u && (p_M\circ v_M = \mathsf{id}_M)\\
&= p_M\circ z_M\\
&= z_M\circ p_M\circ \mathrm{T}p_M\circ [\![u,v]\!]
\end{aligned}$$

□

Definition 4.4 ([Ros84]). Given two vector fields u and v on an object M in a tangent category with negatives, the **Lie bracket** of u and v is the morphism:

$$[u,v] := \{[\![u,v]\!]\}$$

Theorem 4.5 ([CC15, Theorem 4.2]). *In a tangent category with negatives, for each three vector fields u,v,w on an object M:*

(a) $[u,v]$ *is a vector field;*

(b) the Lie bracket is antisymmetric, that is, $[u,v]+[v,u]=0$*;*

(c) The Lie bracket satisfies the Jacobi identity, that is, $[u,[v,w]]+[w,[u,v]]+[v,[w,u]]=0$.

In particular, the Abelian group $\mathsf{VF}(\mathbb{X},\mathbb{T};M)$ *equipped with* [,] *defines a Lie algebra.*

Proving that the Lie bracket of vector fields satisfies the Jacobi identity was a tough goal: there is a whole paper entirely dedicated only to prove that [CC15].

Vector fields of a tangent category $(\mathbb{X}, \mathbb{T})$ form a new tangent category denoted by $\mathsf{VF}(\mathbb{X}, \mathbb{T})$. The objects of $\mathsf{VF}(\mathbb{X}, \mathbb{T})$ are pairs (M, v) formed by an object M of $(\mathbb{X}, \mathbb{T})$ together with a vector field v on M. A morphism $f\colon (M, v) \to (N, u)$ is a morphism $f\colon M \to N$ satisfying the following condition:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\mathrm{T}f} & \mathrm{T}N \\ {\scriptstyle v}\uparrow & & \uparrow{\scriptstyle u} \\ M & \xrightarrow[f]{} & N \end{array}$$

The tangent bundle functor T^{VF} sends an object (M, v) to $(\mathrm{T}M, v_{\mathrm{T}})$, where:

$$v_{\mathrm{T}}\colon M \xrightarrow{\mathrm{T}v} \mathrm{T}\mathrm{T}M \xrightarrow{c_M} \mathrm{T}\mathrm{T}M$$

Moreover, T^{VF} sends a morphism f to $\mathrm{T}f$. Finally, the structural natural transformations are defined as in the base tangent category $(\mathbb{X}, \mathbb{T})$.

Vector fields in the trivial tangent category

In the trivial tangent structure over a category $\mathbb{X}$, the tangent bundle of an object M is M itself and the projection $p_M\colon M \to M$ is the identity. Thus, there is exactly only one vector field: the identity over M. This means that, in the trivial tangent structure, the only available direction is the zero direction.

Vector fields in differential geometry

In the tangent category SMAN, vector fields are exactly vector fields in the traditional sense.

Vector fields in the tangent category of commutative algebras

In the tangent category $(\mathsf{CALG}, \mathbb{T}^{\varepsilon})$, a vector field over an object A is an algebra morphism $v\colon A \to A[\varepsilon]$ such that, $p^{\varepsilon}_A \circ v = \mathsf{id}_A$. Recall that p^{ε}_A sends $a + u\varepsilon$ to a. So, v must send each $a \in A$ to $v_1(a) + v_2(a)\varepsilon$, however, since v is a section of the projection, $v_1(a) = a$. Let us call, $\delta_v(a) := v_2(a)$. Furthermore, v is an algebra homomorphism, therefore, $v(ab) = v(a)v(b)$:

$$ab + \delta_v(ab)\varepsilon = v(ab) = v(a)v(b) = (a + \delta_v(a)\varepsilon)(b + \delta_v(b)) = ab + (a\delta_v(b) + b\delta_v(a))\varepsilon + \delta_v(a)\delta_v(v)\varepsilon^2$$

However, since $\varepsilon^2 = 0$, $\delta_v\colon A \to A$ becomes a derivation of A. It turns out that if $\delta\colon A \to A$ is a derivation of A, then we can define a vector field $v_\delta\colon A \to A[\varepsilon]$ by $v_\delta(a) := a + \delta(a)\varepsilon$. So, vector fields in $(\mathsf{CALG}, \mathbb{T}^{\varepsilon})$ are equivalent to derivations.

Vector fields in algebraic geometry

Now, consider the tangent category $(\mathsf{CALG}^{\mathsf{op}}_R, \mathbb{T}^{\Omega})$ and an object A. A vector field over A corresponds to an algebra homomorphism $\mathrm{T}^{\Omega}A \to A$ (remember that we are in the opposite category) satisfying $v \circ p^{\Omega}_A = \mathsf{id}_A$, where $p^{\Omega}_A\colon A \to \mathrm{T}^{\Omega}A$, as an algebra homomorphism. Since T^{Ω} is generated by each a and $\mathsf{d}a$, for $a \in A$, v is fully specified by the value it takes on each a and on each $\mathsf{d}a$. However, since v is a section of the projection, the value of v at a must be just a. So, v is fully specified by the value $\delta_v(a) := v(\mathsf{d}a)$. However, $\mathsf{d}(ab) = a\mathsf{d}b + b\mathsf{d}a$, therefore, $\delta_v\colon A \to A$ becomes a derivation. Furthermore, if δ is a derivation, then, $v_\delta(a) := a$ and $v_\delta(\mathsf{d}a) := \delta(a)$ fully specifies a vector field. So, vector fields in $(\mathsf{CALG}^{\mathsf{op}}_R, \mathbb{T}^{\Omega})$ are equivalent to derivations.

4.2 Differential objects

Differential objects in a tangent category are the analogue of Euclidean spaces in differential geometry. A Euclidean space consists of a geometric space A equipped with a commutative monoid structure $(A, 0, +)$. Furthermore, Euclidean spaces have an important unique property: the tangent space $\mathrm{T}_x A$ at a given point x of A is isomorphic to A itself, that is, $\mathrm{T}_x A \cong A$. This means, that $\mathrm{T}A \cong A \times A$. This is precisely the "linearity" condition we imposed to the fibres of the tangent bundle $\mathbf{p}_M$ when we constructed the definition of a tangent category.

Notice that the sum $+$ is defined on the product $A \times A$, so, it is natural to require the existence of all binary products.

Definition 4.6 ([CC14, Definition 2.8])**.** A **Cartesian tangent category** is a tangent category with finite products preserved by the tangent bundle functor.

Definition 4.7 ([CC18, Definition 3.1])**.** A **differential object** in a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$ consists of an object A equipped with two morphisms $0\colon * \to A$ and $+\colon A \times A \to A)$, respectively called the zero and the sum of A and a map $\hat{p}\colon \mathrm{T}A \to A$, called the **differential projection** of A, subject to the following conditions:

[DO.1]: $(A, 0, +)$ is a commutative monoid;

[DO.2]: $\hat{p}\colon \mathrm{T}(A, +, 0) \to (A, +, 0)$ is a homomorphism of monoids, where $\mathrm{T}(A, +, 0) := (\mathrm{T}A, \mathrm{T}+, \mathrm{T}0)$ is a commutative monoid since $\mathrm{T}(A \times A) \cong \mathrm{T}A \times \mathrm{T}A$;

[DO.3]: $(!, \hat{p})\colon \mathbf{p}_A \to (!_A, 0, +)$ is an additive bundle morphism, where $!_A\colon A \to *$ is the terminal map;

[DO.4]: $\hat{p}$ is linear, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{TT}A & \xrightarrow{\mathrm{T}\hat{p}} & \mathrm{T}A \\ {\scriptstyle l_A}\uparrow & & \downarrow{\scriptstyle \hat{p}} \\ \mathrm{T}A & \xrightarrow[\hat{p}]{} & A \end{array}$$

[DO.5]: A is linear, that is, the following diagram

$$A \xleftarrow{p_A} \mathrm{T}A \xrightarrow{\hat{p}} A$$

is a product diagram.

Differential objects of a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$ form a new Cartesian tangent category denoted by $\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$. Objects of $\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$ are differential objects and morphisms $f\colon (A, 0_A, +_A, \hat{p}_A) \to (B, 0_B, +_B, \hat{p}_B)$ are morphisms $f\colon A \to B$ of $\mathbb{X}$ (no other conditions are required on f). The tangent bundle functor T^{DO} sends a differential object $(A, 0, +, \hat{p})$ to $(\mathrm{T}(A, 0, +), \hat{p}_{\mathrm{T}})$, where $\mathrm{T}(A, 0, +)$ is the commutative monoid $(\mathrm{T}A, \mathrm{T}0, \mathrm{T}+)$ where we identified $\mathrm{T}(A \times A) \cong \mathrm{T}A \times \mathrm{T}A$, and $\hat{p}_{\mathrm{T}}$ is defined as follows:

$$\hat{p}_{\mathrm{T}}\colon \mathrm{TT}A \xrightarrow{c_A} \mathrm{TT}A \xrightarrow{\mathrm{T}\hat{p}} \mathrm{T}A$$

Furthermore, T^{DO} sends a morphism f to $\mathrm{T}f$. Finally, the structural natural transformations are defined as in the base tangent category $(\mathbb{X}, \mathbb{T})$.

$\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$ comes equipped with a special operation called the **differential combinator**, which associates to every morphism $f\colon A \to B$ a new morphism $\mathsf{D}[f]\colon A \times A \to B$ defined as follows:

$$\mathsf{D}[f]\colon A \times A \xrightarrow{\langle \pi_1, \pi_2 \rangle} \mathrm{T}A \xrightarrow{\mathrm{T}f} \mathrm{T}B \xrightarrow{\hat{p}_B} B$$

where we used that $A \times A \cong \mathrm{T}A$ from the universal property of the differential projection $\hat{p}_A$ of A. It turns out that D makes $\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$ into a Cartesian differential category (CDC, see the section of the world of differential categories). Moreover, every CDC is a tangent category whose objects are all differential objects.

In $\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$, a morphism $f \colon A \to B$ is **linear** provided that f commutes with the differential projections as follows:

$$\begin{array}{ccc} \mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\ {\scriptstyle \hat{p}_A}\downarrow & & \downarrow{\scriptstyle \hat{p}_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

This condition translates in the following equation:

$$\mathsf{D}[f] = f \circ \pi_2$$

This means that f is linear if and only if it coincides with its first derivative at 0: $f(x) = f'(0) \cdot x$. The subcategory of $\mathsf{DO}^\star(\mathbb{X}, \mathbb{T})$ spanned by linear morphisms is also a sub-tangent category denoted by $\mathsf{DO}(\mathbb{X}, \mathbb{T})$.

We postpone the examples for later.

4.3 The problem with pullbacks

This section is devoted to solve once for all the issue with pullbacks in a tangent category. The reader can ignore this section, since it is a bit technical. Well, it is technical, because it solves an important but technical problem! The problem is the following one. In the category SMAN pullbacks in general do not exist and when they do, the tangent bundle functor might still not preserve them. Geoff Cruttwell and I wrote a whole paper on this issue [CL25]. Our solution is to construct a family of maps that behave well with respect to pullbacks and the tangent bundle functor. First of all, let us make precise the definition of a tangent pullback.

Definition 4.8. A **tangent limit** of a diagram $D \colon I \to \mathbb{X}$ in a tangent category $(\mathbb{X}, \mathbb{T})$ is a limit cone $\lambda_i \colon L \to D(i)$, for $i \in I$, of D, preserved by all iterates of the tangent bundle functor, that is, such that for each $m \geq 0$, $\mathrm{T}^m\lambda_i \colon \mathrm{T}^m L \to \mathrm{T}^m D(i)$ is a limit cone for the diagram $\mathrm{T}^m \circ D \colon I \to \mathbb{X}$.

A **tangent pullback** is a pullback that, as a limit cone, is a tangent limit.

Definition 4.9. A **tangent display map** in a tangent category $(\mathbb{X}, \mathbb{T})$ is a map $q \colon E \to M$ such that, for every $n \geq 0$, and for every $f \colon M' \to \mathrm{T}^n M$, the pullback of $\mathrm{T}^n q$ along f exists and is a tangent pullback.

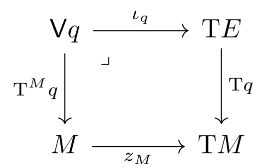
We proved that, in the tangent category SMAN of smooth manifolds, tangent display maps are exactly the submersions [CL25, Theorem 2.31]. We shall assume that the projection $p_M \colon \mathrm{T}M \to M$ is a tangent display map.

4.4 The slice tangent category and the axiom of local linearity

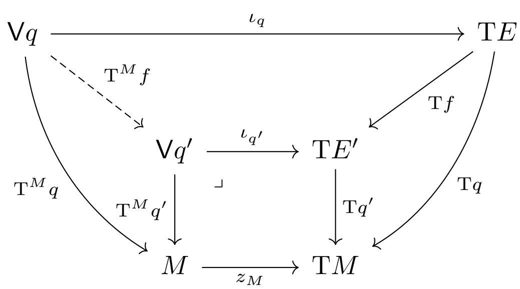
In the previous sections, we have seen how to define spaces that are "linear". Now, we can review the axiom **[TAN.7]** of local linearity in Definition 3.5. To justify of adding the vertical lift in the definition of a tangent category and to force on the objects in a tangent category to exhibit a local linear behaviour, we required the tangent bundle $\mathbf{p}_M \colon \mathrm{T}M \to M$ to be a "linear object" in the slice category $\mathbb{X}/M$ with respect to the functor $\mathrm{T}^M \colon \mathbb{X}/M \to \mathbb{X}/M$. Now, we want to make this idea precise. The first step is to construct a proper tangent structure on $\mathbb{X}/M$. We call this, the **slice tangent category** over M.

We start by recalling that the slice category $\mathbb{X}/M$ over an object M is the category whose objects are *tangent display maps* $q \colon E \to M$ and morphisms $f \colon q \to q'$ are morphisms $f \colon E \to E'$ that commute with $q \colon E \to M$ and $q' \colon E' \to M$, that is, $q' \circ f = q$. Notice that, usually for $\mathbb{X}/M$ one refers to the category whose objects are all maps of type $q\colon E \to M$, however, since we need to deal with pullbacks frequently, we better restrict ourselves from the beginning to only the tangent display maps.

Next, we may want to recall the construction of the tangent bundle functor $\mathrm{T}^M\colon \mathbb{X}/M \to \mathbb{X}/M$. For an object $q\colon E \to M$ of $\mathbb{X}/M$, $\mathrm{T}^M q$ is the map in $\mathbb{X}$ defined as the pullback of $\mathrm{T}q$ along z_M:



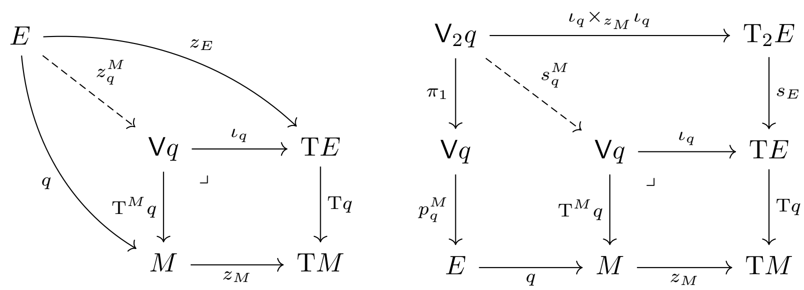
Now, consider a morphism $f\colon q \to q'$ of $\mathbb{X}/M$ and define $\mathrm{T}^M f\colon \mathrm{T}^M q \to \mathrm{T}^M q'$ as the unique morphism that renders the following diagram commutative:



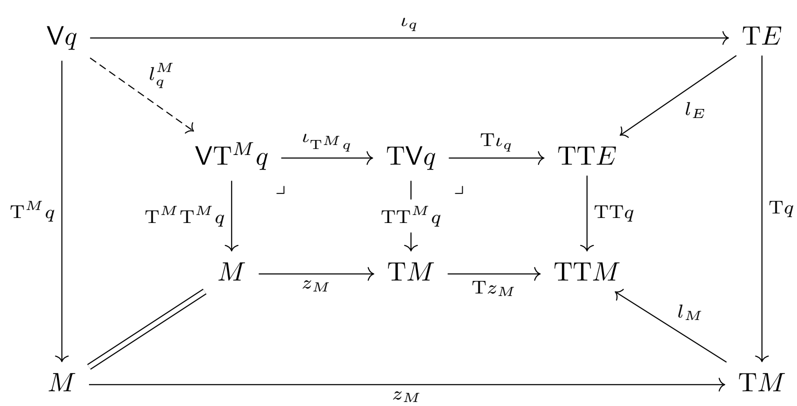
It turns out that T^M is in fact a functor. Next, the projection $p^M_q\colon \mathrm{T}^M q \to q$ is given by the morphism:

$$p^M_q\colon \mathsf{V}q \xrightarrow{\iota_q} \mathrm{T}E \xrightarrow{p_E} E$$

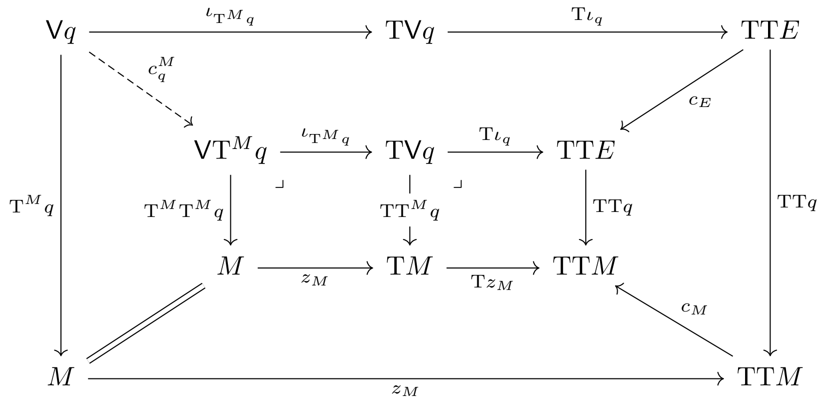
The zero $z^M_q\colon q \to \mathrm{T}^M q$ and the sum $s^M_q\colon \mathrm{T}^M_2 q \to \mathrm{T}^M q$ are defined as follows:



The vertical lift $l^M_q\colon \mathrm{T}^M q \to \mathrm{T}^M\mathrm{T}^M q$ is defined as follows:



Finally, the canonical flip $c_q^M \colon \mathrm{T}^M\mathrm{T}^M q \to \mathrm{T}^M\mathrm{T}^M q$ is defined as follows:



Proposition 4.10 ([Ros84])**.** $\mathbb{T}^M := (\mathrm{T}^M, p^M, z^{\cdot}s^M, l^M, c^M)$ *is a tangent structure on the slice category* $\mathbb{X}/M$*.*

Recall that we are assuming that the projection $p_M \colon \mathrm{T}M \to M$ to be a tangent display map. This is a fairly general assumption that covers all the important examples.

Theorem 4.11. *Given a tangent category* $(\mathbb{X}, \mathbb{T})$*, for each object* M*, the tangent bundle* $\mathbf{p}_M \colon \mathrm{T}M \to M$ *is a differential object in the slice tangent category* $(\mathbb{X}/M, \mathbb{T}^M)$*.*

Proof. As observed in Lemma 3.2, every additive bundle is precisely a commutative monoid in the slice category, so in particular, the tangent bundle $\mathbf{p}_M$ defines a commutative monoid in $\mathbb{X}/M$. Next, thanks to the universal property of the vertical lift, $\mathrm{T}^M p_M$, which is the pullback of Tp_M along z_M, is isomorphic to $\mathrm{T}_2 M \to M$, however, $\mathrm{T}_2 M = \mathrm{T}M \times_M \mathrm{T}M$, which means that $\mathrm{T}^M p_M \cong p_M \times_M p_M$, that is, the tangent bundle of p_M in the slice tangent category must be isomorphic to the product (in the slice, so the pullback) of p_M with itself. Furthermore, since the isomorphism comes from the vertical lift, since l_M is additive, this isomorphism preserves the commutative monoid structure of $\mathbf{p}_M$. From this observation, it is not hard to construct a projection $\hat{p}_M \colon \mathrm{T}^M p_M \to p_M$, which makes $\mathbf{p}_M$ into a differential object in $(\mathbb{X}/M, \mathbb{T}^M)$. □

4.5 Differential bundles

In the previous section, we showed that the axiom of local linearity, that is, the universal property of the vertical lift, can be translated by saying that the tangent bundle $\mathbf{p}_M$ of each object M in a tangent category, is a differential object in the slice tangent category over M. This characterization open a new question: what are the other differential objects in $(\mathbb{X}/M, \mathbb{T}^M)$? In this section, we introduce an important class of maps, the differential bundles, of a tangent category and show that this class characterizes precisely all the differential objects in the slice. Let's start with a definition.

Definition 4.12 ([CC18, Definition 2.3])**.** A (**display**) **differential bundle** in a tangent category consists of the following morphisms

$$q \colon E \to M \qquad z_q \colon M \to E \qquad s_q \colon E_2 \to E \qquad l_q \colon E \to \mathrm{T}E$$

respectively called, the **projection**, the **zero**, and **sum**, and the **vertical lift**, where E_2 denotes the pullback q along itself, subject to the following conditions:

[DB.1]: $q \colon E \to M$ is a tangent display map;

[DB.2]: $\mathbf{q} := (q, z_q, s_q)$ is an additive bundle;

[DB.3]: The vertical lift l_q is additive, that is, $(z_M, l_q)\colon \mathbf{q} \to \mathrm{T}\mathbf{q}$ and $(z_q, l_q)\colon \mathbf{q} \to \mathbf{p}_E$ are additive bundle morphisms;

[DB.4]: The vertical lift l_q is linear, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}E & \xrightarrow{\mathrm{T}l_q} & \mathrm{T}\mathrm{T}E \\ {\scriptstyle l_q}\uparrow & & \uparrow{\scriptstyle l_E} \\ E & \xrightarrow[l_q]{} & \mathrm{T}E \end{array}$$

[DB.5]: E is **locally linear**, that is, the following diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\xi_q} & \mathrm{T}E \\ {\scriptstyle \pi_1 q}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{T}q} \\ M & \xrightarrow[z_M]{} & \mathrm{T}M \end{array}$$

is a tangent pullback, where:

$$\xi_q\colon E_2 \xrightarrow{\langle l_q, z_E\rangle} \mathrm{T}E_2 \xrightarrow{\mathrm{T}s_q} \mathrm{T}E$$

We shall denote a differential bundle $(q\colon E \to M, z_q, s_q, l_q)$ simply by $\mathbf{q}\colon E \to M$. A differential bundle can be regarded as a generalization of the tangent bundle. **[DB.1]** and **[DB.2]** tells us that $\mathbf{q}$ has the structure of an additive bundle, that is, that the fibres E_x of q have a commutative monoid structure. **[DB.5]** establishes that the fibres E_x are differential objects. We shall make this point more precise in a second. First, let us introduce the notion of morphisms of differential bundles.

Definition 4.13 ([CC18, Definition 2.3])**.** A morphism of differential bundles $(f, g)\colon \mathbf{q} \to \mathbf{q}'$ consists of a pair of morphisms $f\colon M \to M'$ and $g\colon E \to E'$ that commutes with the projections, that is:

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

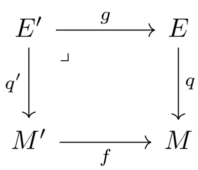
A morphism (f, g) of differential bundles is **linear** provided that g commutes with the vertical lifts, that is:

$$\begin{array}{ccc} \mathrm{T}E & \xrightarrow{\mathrm{T}g} & \mathrm{T}E' \\ {\scriptstyle l_q}\uparrow & & \uparrow{\scriptstyle l'_q} \\ E & \xrightarrow[g]{} & E' \end{array}$$

Theorem 4.14 ([CC18, Propositions 3.4, 5.12])**.** *In a tangent category $(\mathbb{X}, \mathbb{T})$, a differential bundle $\mathbf{q}\colon E \to M$ is equivalent to a differential object in the slice tangent category $(\mathbb{X}/M, \mathbb{T}^M)$ over M. Moreover, when $(\mathbb{X}, \mathbb{T})$ is Cartesian, differential objects of $(\mathbb{X}, \mathbb{T})$ are equivalent to differential bundles over the terminal object.*

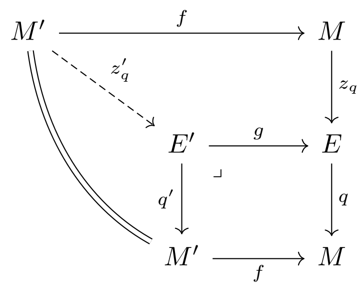
One important property of differential bundles is that, they are stable under tangent pullbacks.

Proposition 4.15 ([CC18, Lemma 2.7])**.** *Consider a morphism $f\colon N \to M$ in a tangent category and a differential bundle $\mathbf{q}\colon E \to M$. Consider the tangent pullback*

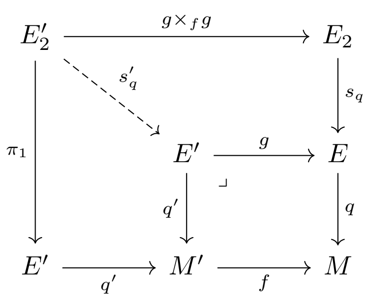


of q along f. Then $q'\colon E' \to M'$ comes with the structure of a differential bundle such that $(f, g)\colon \mathbf{q}' \to \mathbf{q}$ becomes a linear morphism of differential bundles.

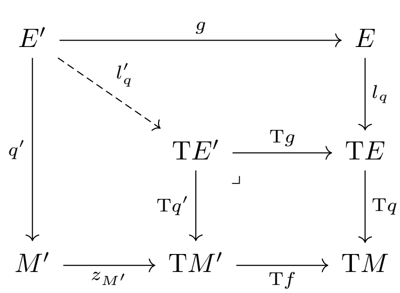
Proof. We sketch the proof by showing how to construct the maps of the differential bundle $\mathbf{q}'$ without actually proving that we obtain the structure of a differential bundle. We start with the zero. To construct $z'_q\colon M' \to E'$, we use the universal property of the pullback as follows



where we used that $q \circ z_q = \mathsf{id}_M$. To construct the sum, we use a similar strategy:



Notice that, tangent display maps are stable under tangent pullbacks, so q' is also tangent display, so in particular, the n-fold tangent pullback E'_n of q' along itself exists. Finally, we shall construct the vertical lift of q' as follows:



Notice that g commutes with the vertical lifts, that is, (f, g) becomes a linear morphism of differential bundles. We leave it to the reader to prove that $\mathbf{q}' := (q', z'_q, s'_q, l'_q)$ is in fact a differential bundle. $\square$

The proposition we just proved has some interesting applications. Before we claimed that the condition of local linearity can be interpeted by saying that the fibres E_x of a differential bundle $\mathbf{q}\colon E \to M$ are differential objects. Now, can we make precise this idea as follows.

Theorem 4.16 ([CC18, Corollary 3.5])**.** *Given a point $x\colon * \to M$ and a differential bundle $\mathbf{q}\colon E \to M$, the* ***fibre*** *of q over x, that is, the tangent pullback*

$$\begin{array}{ccc} E_x & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle q} \\ * & \xrightarrow[x]{} & M \end{array}$$

of q along x, is a differential object. In particular, tangent spaces T_xM are differential objects.

Proof. By Proposition 4.15, the tangent pullback of q along x is a differential bundle, however, this is a differential bundle over the terminal object. Thus, by Theorem 4.14, it is a differential object. Since $\mathbf{p}_M\colon \mathrm{T}M \to M$ is a differential bundle, it follows immediately that tangent spaces are differential objects. ☐

The trivial differential bundle

Every object M in a tangent category admits at least a differential bundle on it, the trivial tangent bundle, denoted by 0_M. The projection, the zero, and the sum of 0_M are the identity morphisms of M and the vertical lift is just $z_M\colon M \to \mathrm{T}M$. The category $\mathsf{DB}(\mathbb{X}, \mathbb{T}; M) \cong \mathsf{DO}((\mathbb{X}, \mathbb{T})/M)$ of differential bundles over M admits biproducts and the zero object of $\mathsf{DB}(\mathbb{X}, \mathbb{T}; M)$ is precisely the trivial differential bundle 0_M of M.

Differential bundles in differential geometry

Ben MacAdam classified differential bundles in the category SMAN [Mac21]. According to this classification, a differential bundle in SMAN is equivalent to a vector bundle. In differential geometry, there are two flavours of vector bundles. Sometimes, a vector bundle is defined as a fibre bundle $q\colon E \to M$ whose typical fibre F comes with the structure of a vector space. Other times, a vector bundle is defined more generally, as a "generalized" fibre bundle, that is, a surjective smooth map $q\colon E \to M$ such that, each point x of M admits an open neighbourhood U such that, $q^{-1}(U) \cong U \times E_x$, where $E_x := q^{-1}(x)$ is the local fibre. In particular, no requirement is made on local fibres to be isomorphic to each other. In this more general sense, a vector bundle is a "generalized" fibre bundle whose local fibres carry the structure of a vector space. When the base object is connected, these two definitions coincide, however, in general, the second version is more general, since disconnected components might have non-isomorphic fibres. According to the classification of MacAdam, differential bundles in SMAN correspond exactly to the more general version of a vector bundle.

With this classification of differential bundles, now it is easy to understand also differential objects. Remember that by Theorem 4.14, differential objects are the same as differential bundles over the terminal objects. The terminal object of SMAN is the sigleton $\{*\}$ and a vector bundle over $\{*\}$ corresponds to a vector space $\mathbb{R}^n$. So, differential objects of SMAN corresponds to Euclidean spaces.

Differential bundles in the tangent category of commutative algebras

In [CL23], Geoff Cruttwell and JS Lemay classified differential bundles in the tangent category $(\mathsf{CALG}_R, \mathbb{T}^\varepsilon)$. A differential bundle in that tangent category corresponds to a projection $q\colon B \to A$, with a zero $A \to B$, a sum $B_2 \to B$, and a lift $B \to B[\varepsilon]$. It turns out that every additive bundle in CALG_R is always (isomorphic to an additive bundle) of the form $A \ltimes M \to A$, where M is an A-module, where $M := \mathsf{ker}\, q$ and where $A \ltimes M$ is the

R-algebra of elements $a + x\varepsilon$, with $a \in A$ and $x \in M$, such that $\varepsilon^2 = 0$. Equivalently, $A \ltimes M$ is the R-algebra of pairs $(a, x) \in A \times M$, where the multiplication is defined by:

$$(a, x) \cdot (b, y) := (ab, ay + bx)$$

and the unit is given by $(1, 0)$. It turns out that, the vertical lift is not adding anything (in this special tangent category) and that every additive bundle is already a differential bundle. So, this the tangent category $(\mathsf{CALG}_R, \mathbb{T}^\varepsilon)$, differential bundles are equivalent to modules over R-algebras, that is, $\mathsf{DB}(\mathsf{CALG}_R, \mathbb{T}^\varepsilon) \simeq \mathsf{MOD}$.

Using a similar strategy as before, we can classify differential objects. Notice that, in CALG_R, the terminal object is the zero R-algebra 0, however, there is no 0-module, rather than the trivial one. So, the only differential object of $(\mathsf{CALG}_R, \mathbb{T}^\varepsilon)$ is 0.

Differential bundles in algebraic geometry

In [CL23], Geoff Cruttwell and JS Lemay also classified differential bundle in the tangent category $(\mathsf{CALG}_R^{\mathsf{op}}, \mathbb{T}^\Omega)$. Surprisingly, also in this case, differential bundles correspond to modules. In this case, there is an equivalence $\mathsf{DB}(\mathsf{CALG}_R^{\mathsf{op}}, \mathbb{T}^\Omega) \simeq \mathsf{MOD}^{\mathsf{op}}$. Concretely, every differential bundle in $(\mathsf{CALG}_R^{\mathsf{op}}, \mathbb{T}^\Omega)$ consists of a map $q \colon A \to B$ (regarded as a morphism in CALG_R) together with a zero $z_q \colon B \to A$, a sum $B \to B_2$, where $B = B \otimes_A B$, and a lift $\mathrm{T}^\Omega B \to B$. It turns out that, B is always (isomorphic to an R-algebra) of the form $\mathsf{Sym}_A M$, for an A-module M. In this tangent category, additive bundles are not necessarily differential bundles, since in an additive bundle, one might have "higher order" terms, such as in $A \oplus M_1 \oplus (M_2 \otimes M_2) \oplus \dots$. Instead, the universality of the vertical lift forces the "higher order" terms to be completely determined by the linear terms, which means that B must be of the form $A \oplus M \oplus (M \otimes M) \oplus \dots = \mathsf{Sym}_A M$.

In the tangent category of schemes, differential bundles correspond to quasi-coherent sheaves of modules.

Using the same strategy as before, we also classify differential objects for $(\mathsf{CALG}_R^{\mathsf{op}}, \mathbb{T}^\Omega)$. The terminal object of $\mathsf{CALG}_R^{\mathsf{op}}$ is the initial object in CALG_R, that is, the R-algebra R. Therefore, differential objects in $(\mathsf{CALG}_R^{\mathsf{op}}, \mathbb{T}^\Omega)$ correspond to R-modules under the equivalence $B \cong \mathsf{Sym}_R M$, that is, differential objects are precisely the free R-algebras.

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