Dalhousie University
Mathematics & Statistics

The Grothendieck construction in the context of Tangent categories

Marcello Lanfranchi

Supervisors: Dorette Pronk & Geoffrey Cruttwell

Tangent categories and tangent fibrations

chapter 1

"The objects of a tangent category
are locally linear spaces.

Tangent fibrations are
fibrations of tangent categories."

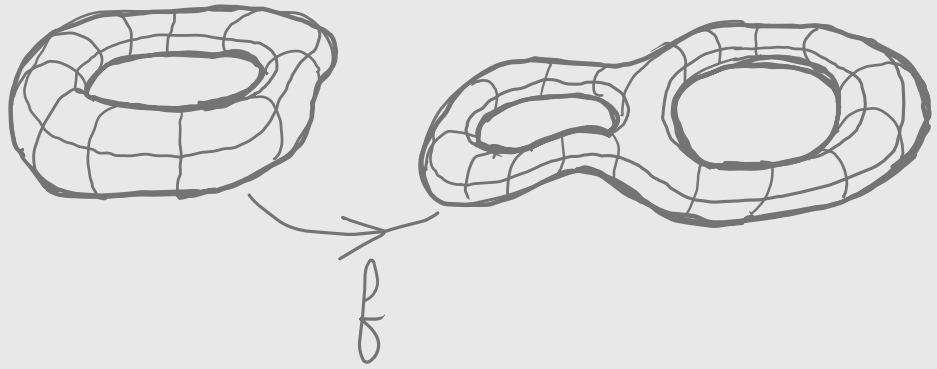
Tangent categories and tangent fibrations

Rosicky 1984 | Cockett Cruttwell 2014

definition

A **tangent category** consists of:

A category
Tangent bundle functor
Projection
Zero morphism
Sum morphism
Vertical lift
Canonical flip



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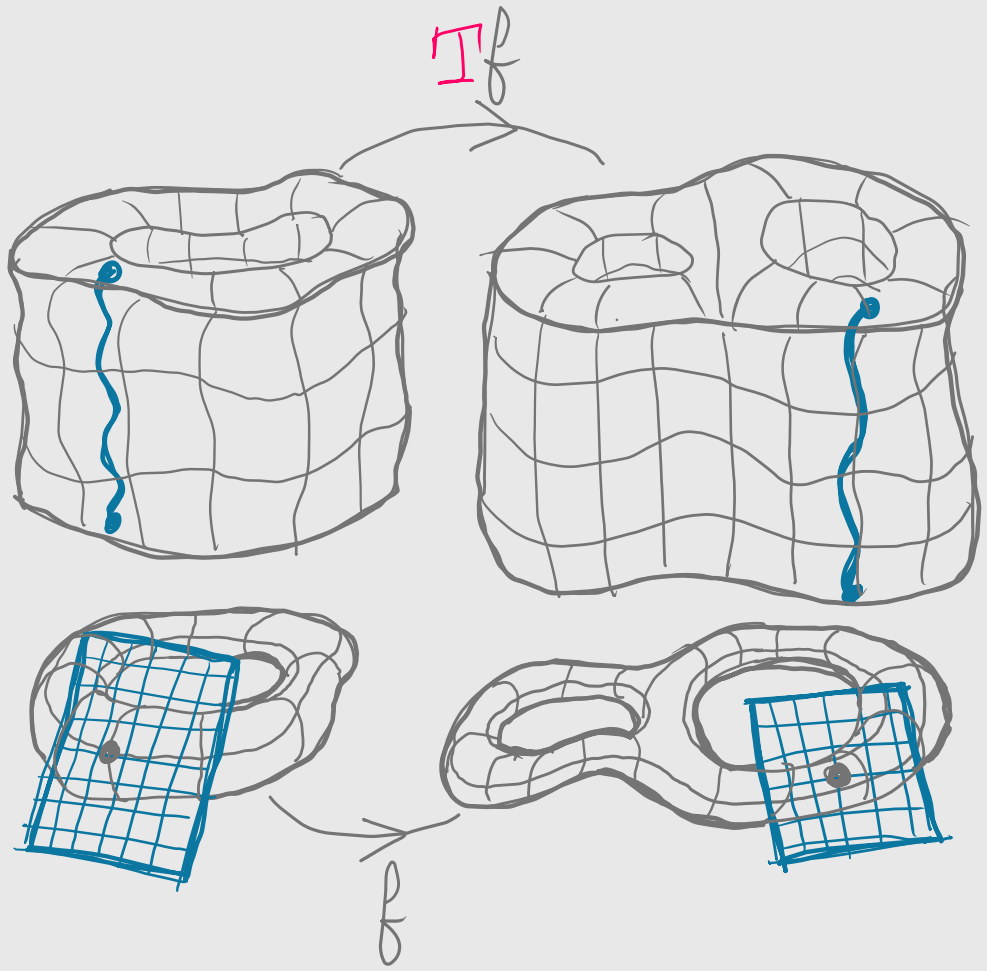

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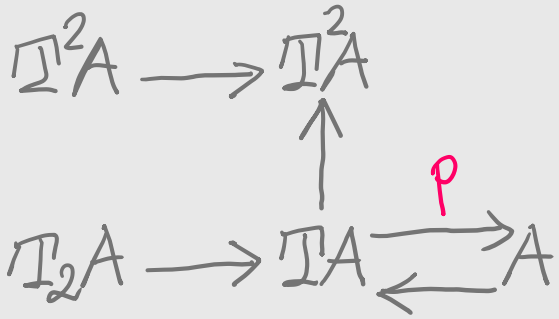

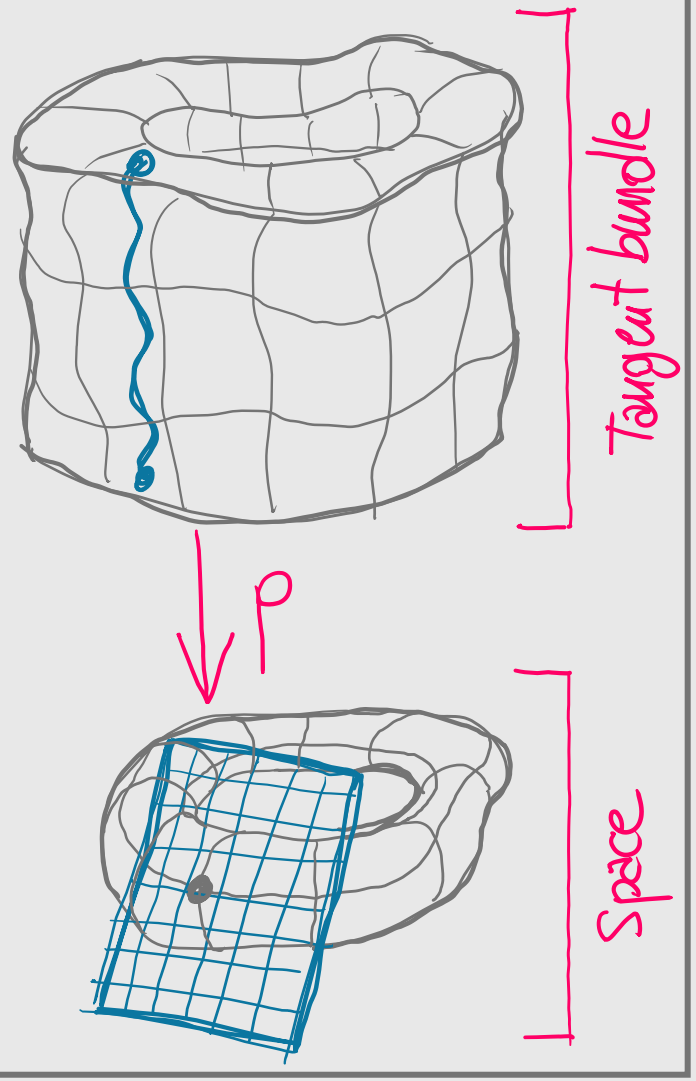

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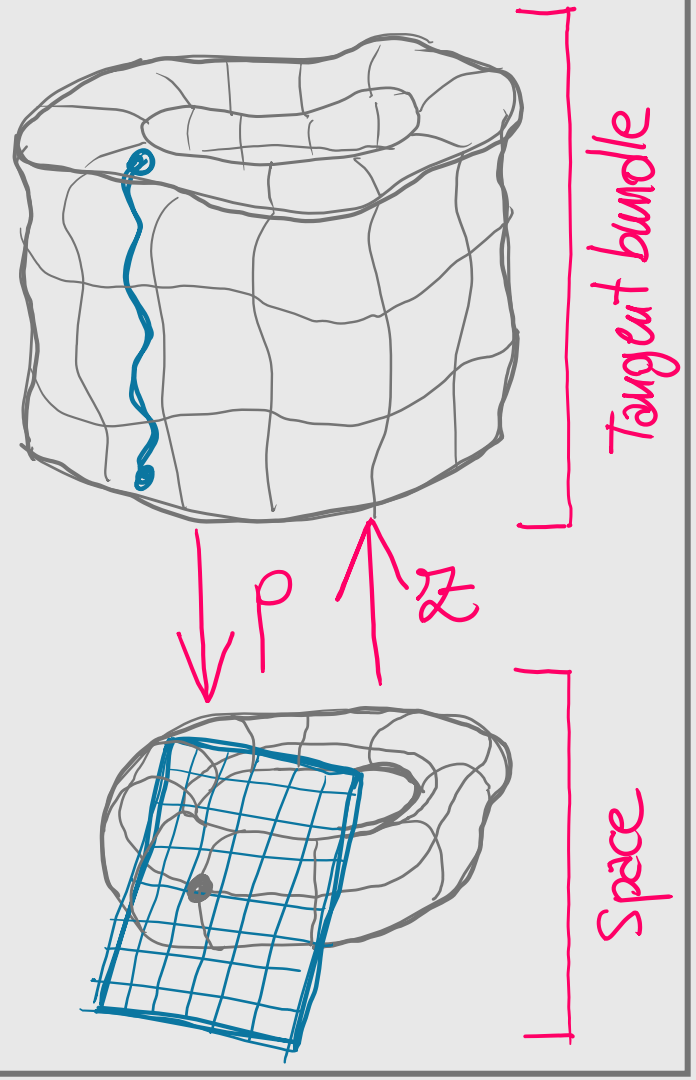
$$\begin{array}{ccc} T^2A & \longrightarrow & T^2A \\ & & \uparrow \\ T_2A & \longrightarrow & TA \underset{z}{\overset{p}{\rightleftarrows}} A \end{array}$$



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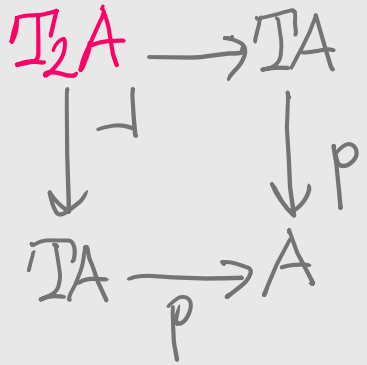

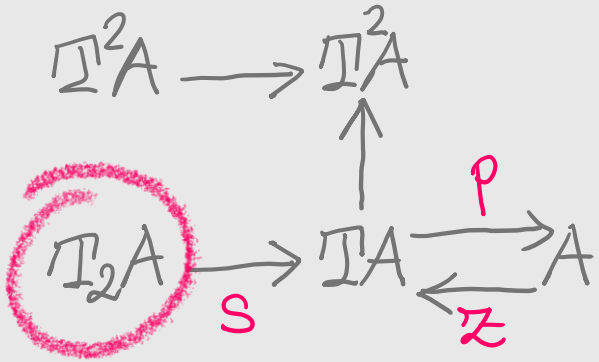

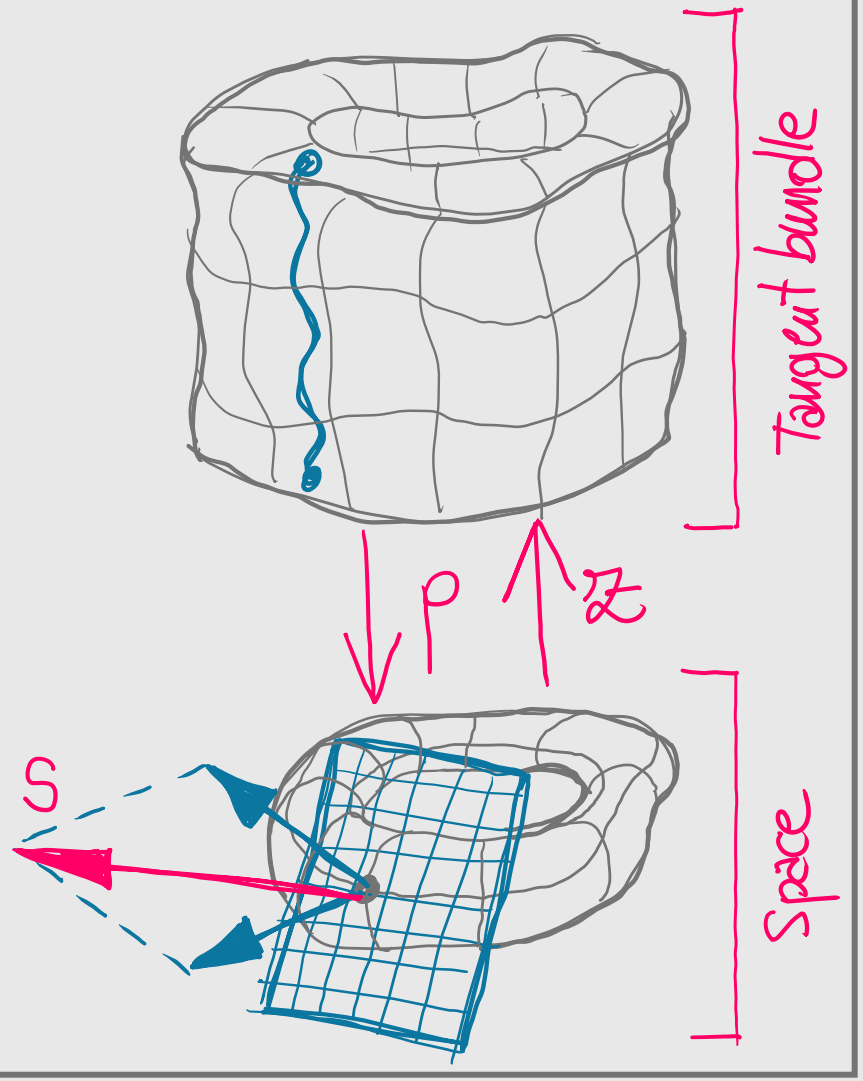

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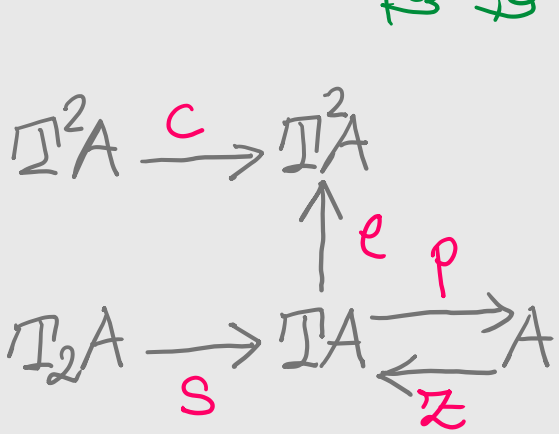

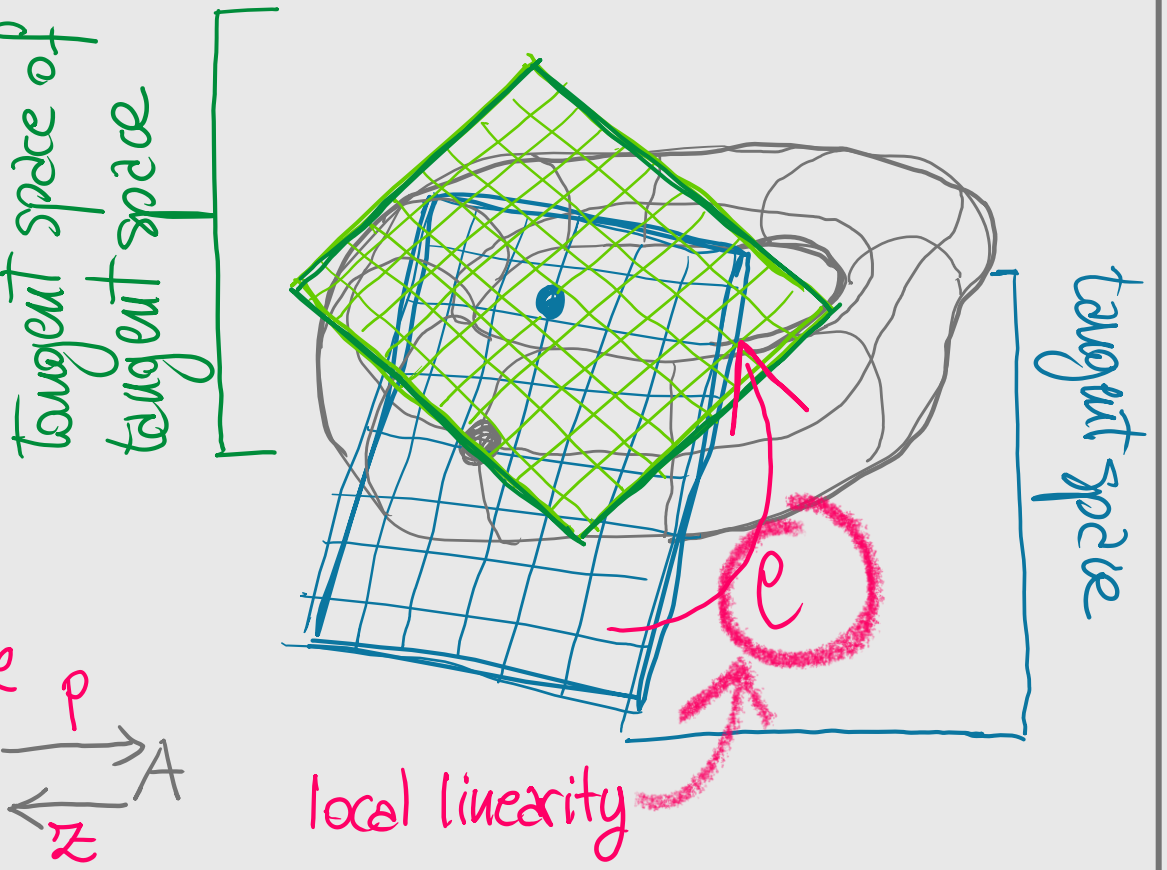

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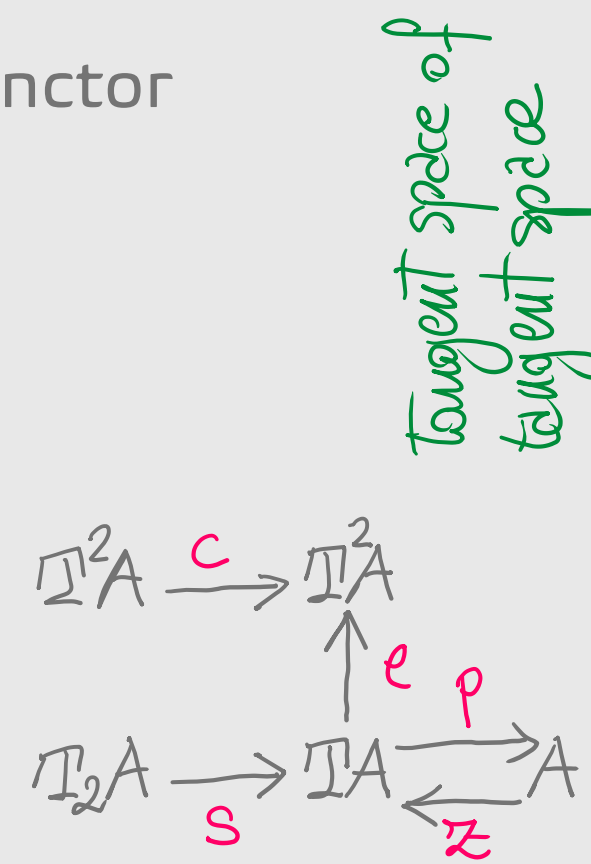

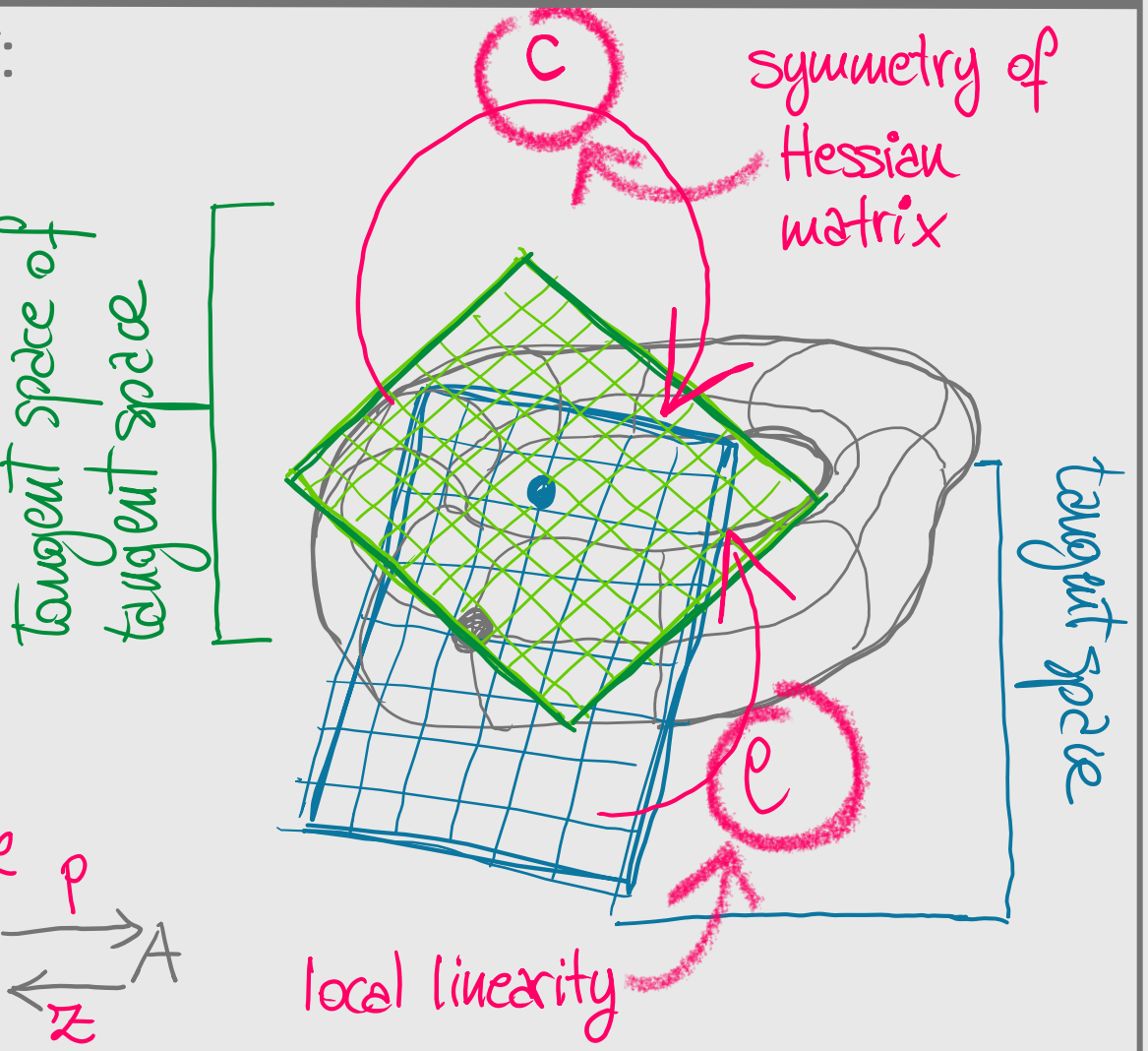

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A (**cloven**) **tangent fibration** is:

a (cloven) fibration between
two tangent categories which
preserves strictly
the tangent structures.

The tangent bundle functors
preserve the cartesian lifts.

Tangent categories and tangent fibrations

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2017

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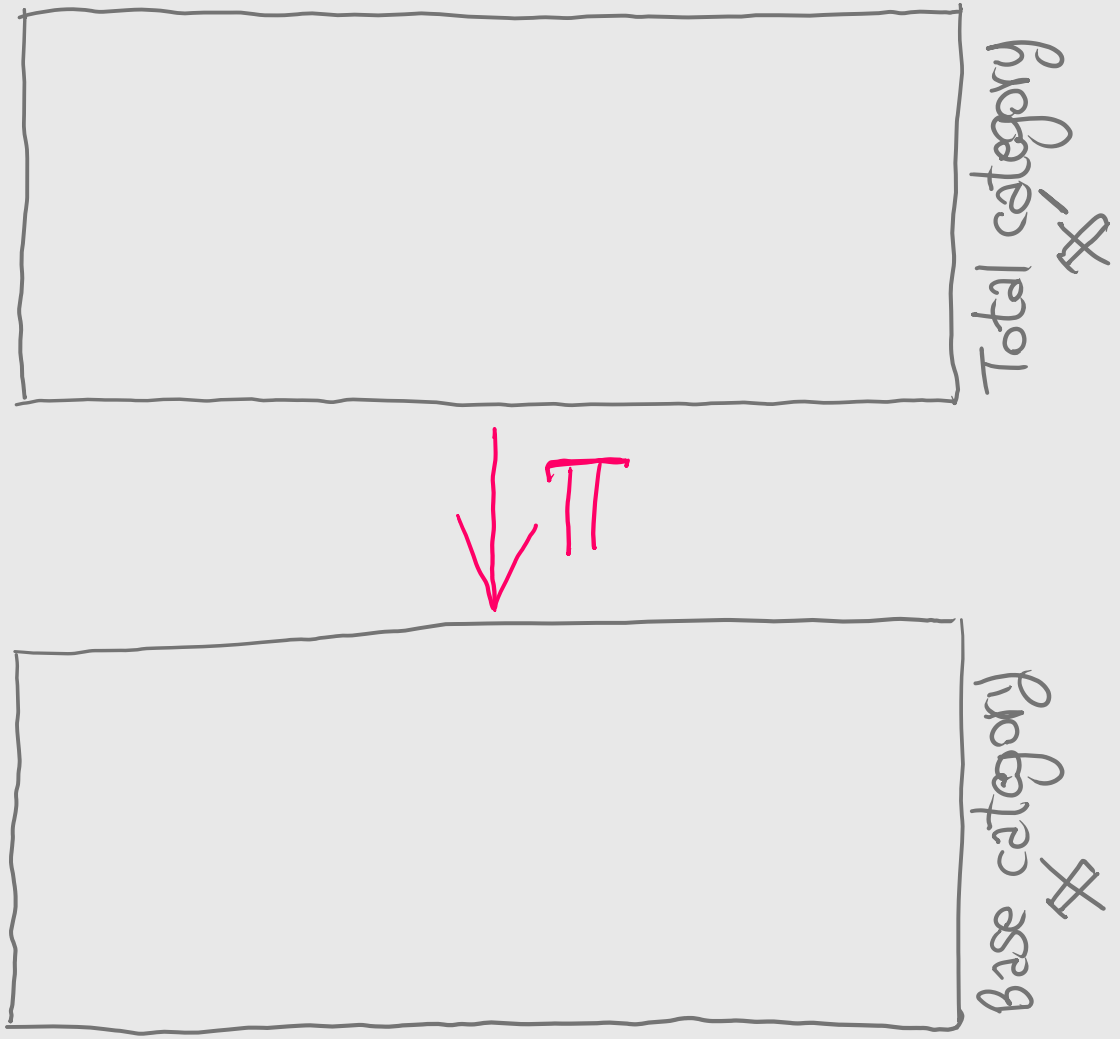

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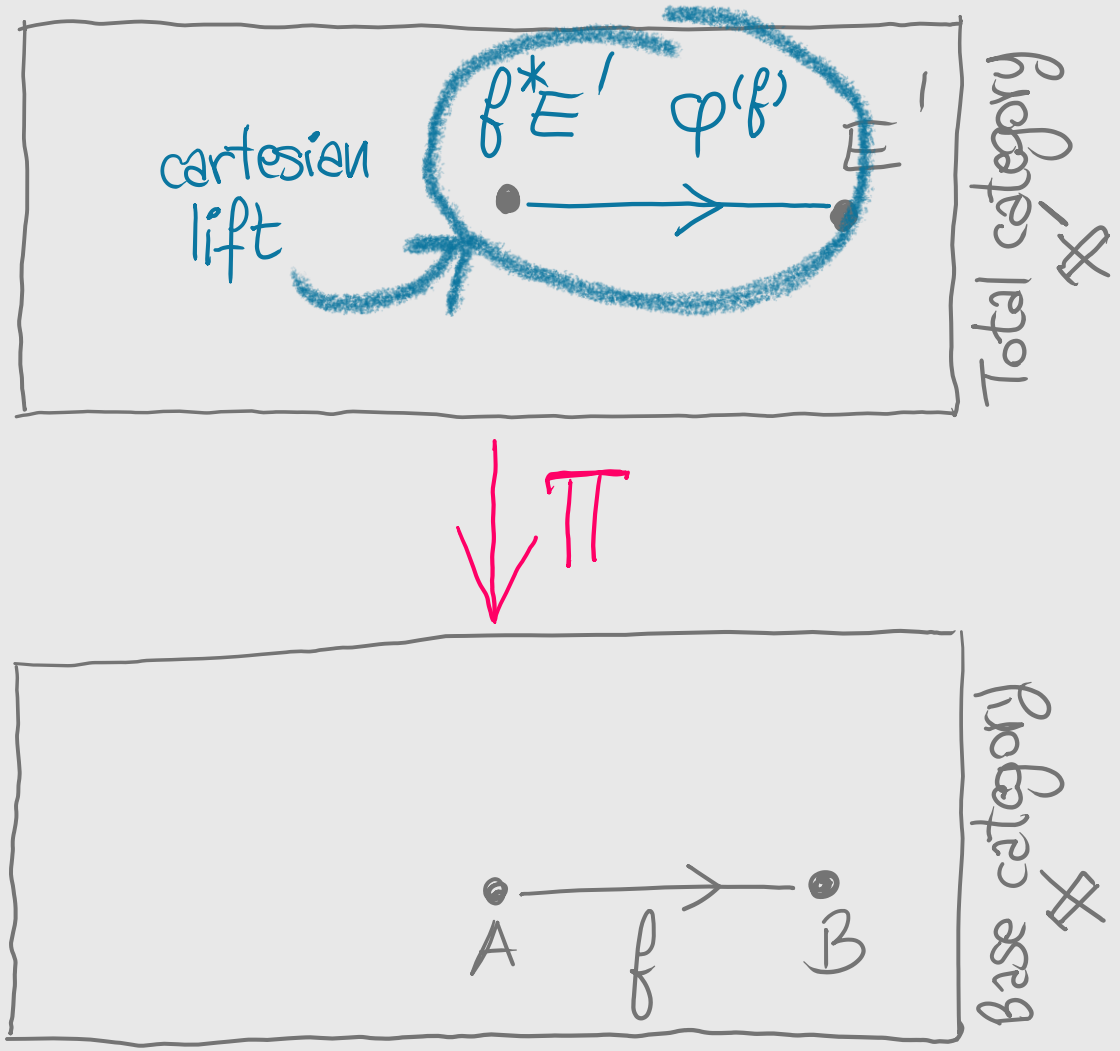

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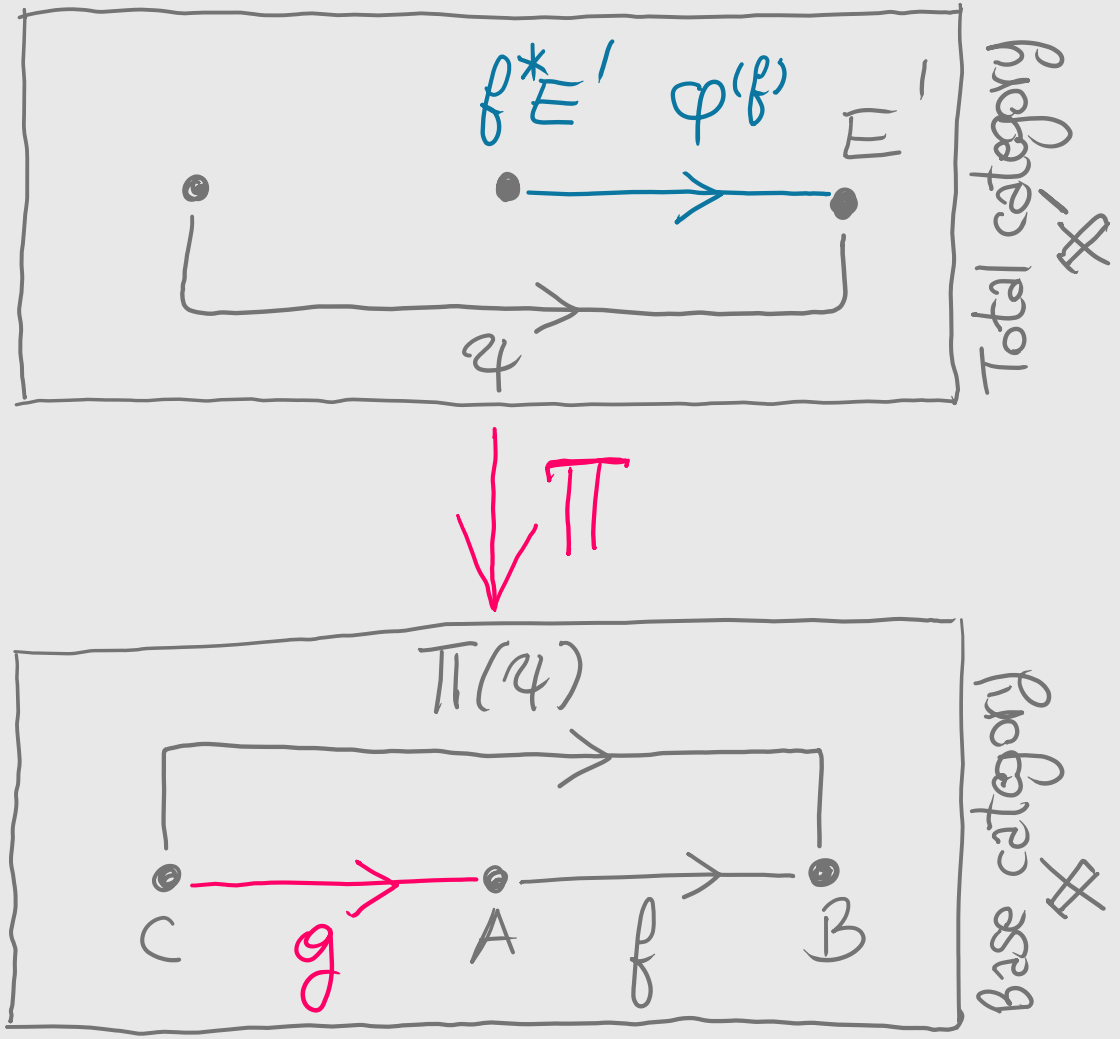

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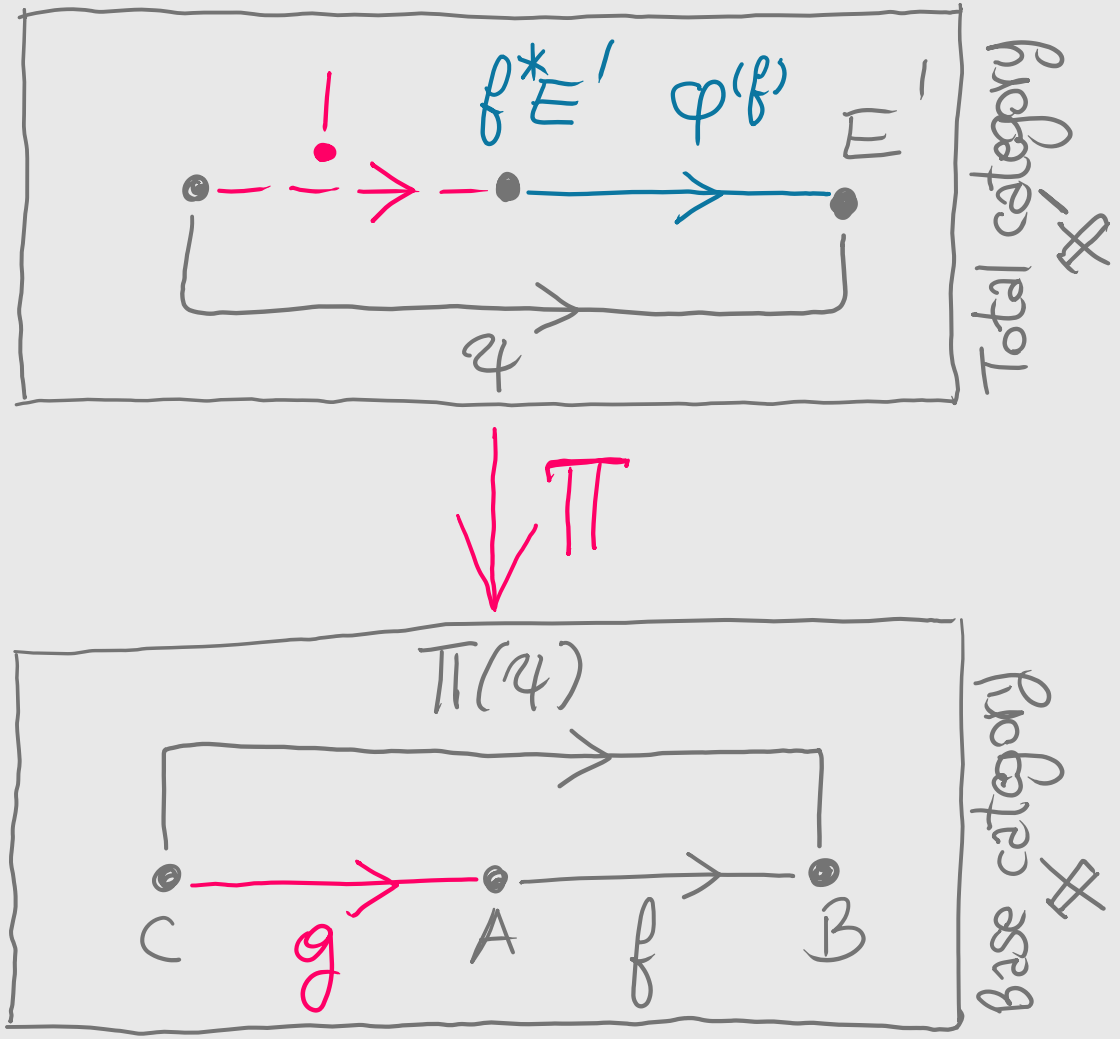

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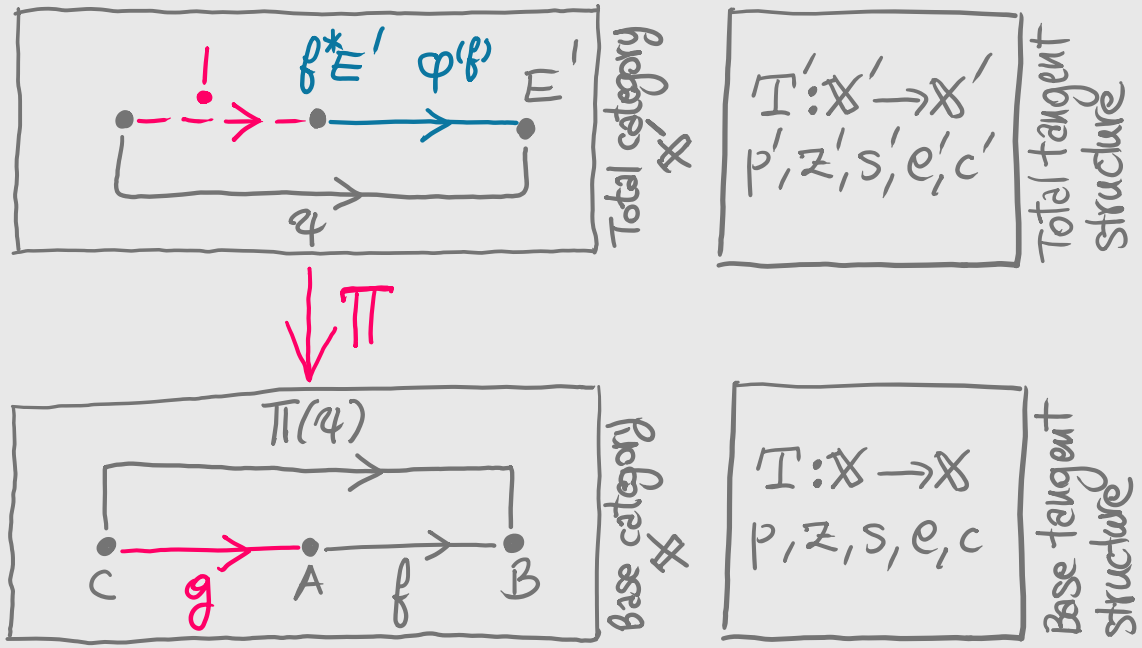

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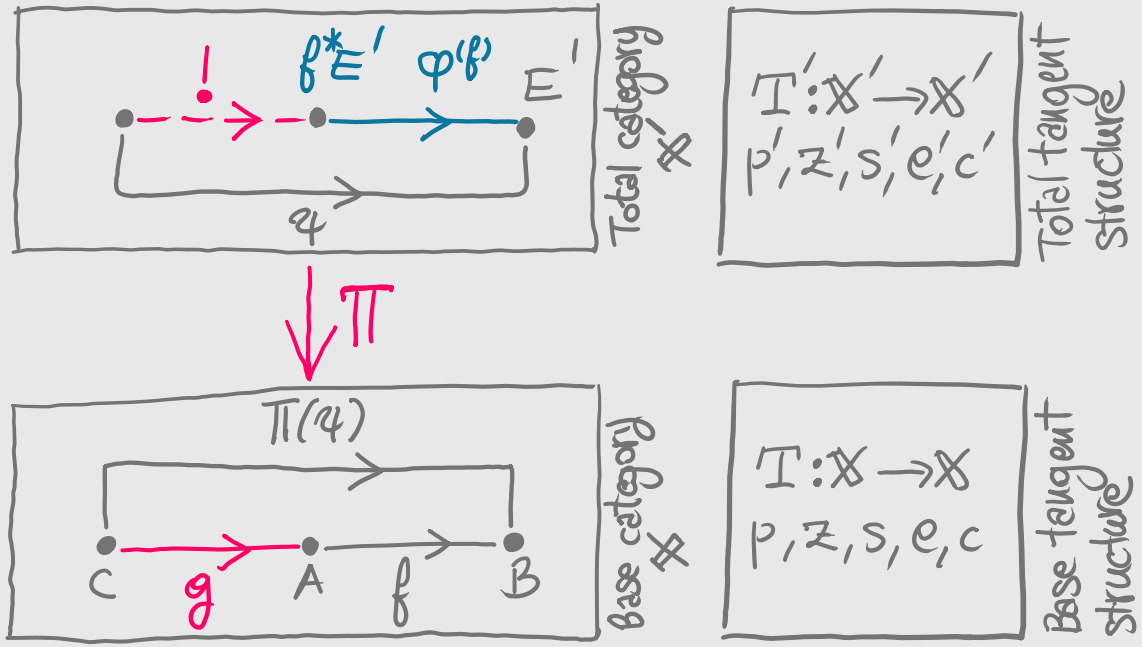

$$\pi\mathbb{T}'=\mathbb{T}\pi \qquad \pi s'=s_\pi$$
$$\pi p'=p_\pi \qquad \pi e'=e_\pi$$
$$\pi z'=z_\pi \qquad \pi c'=c_\pi$$

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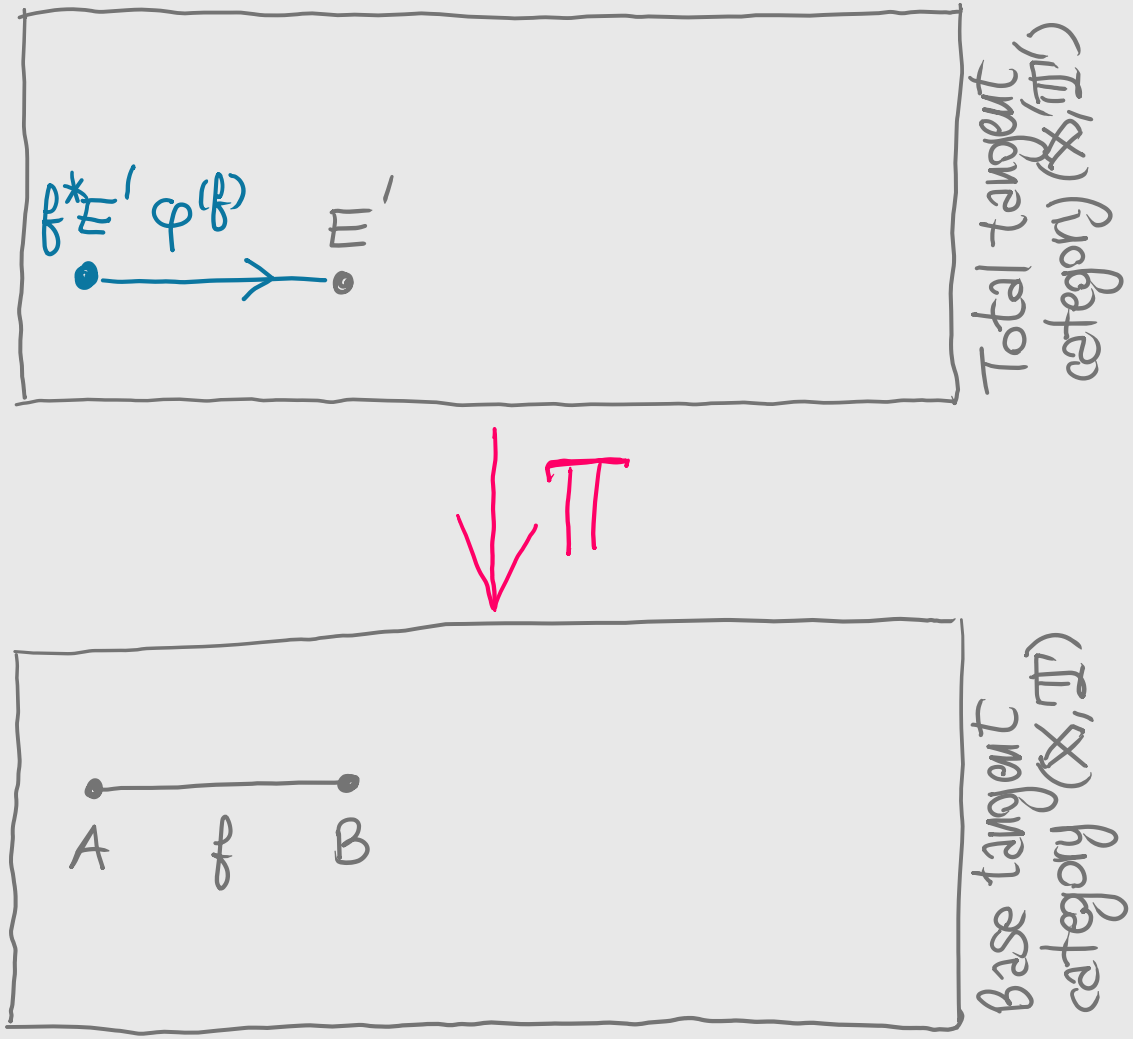

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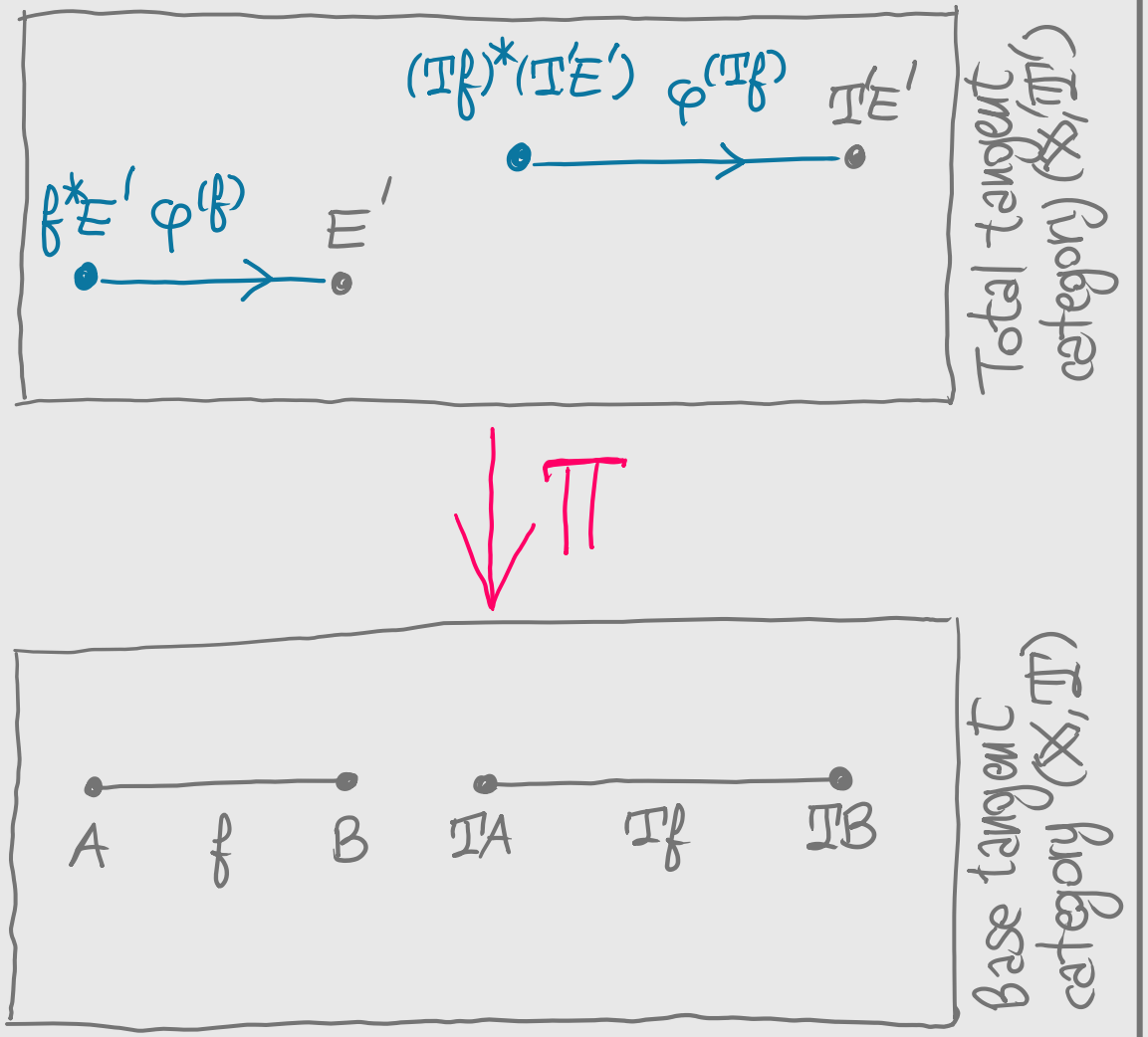

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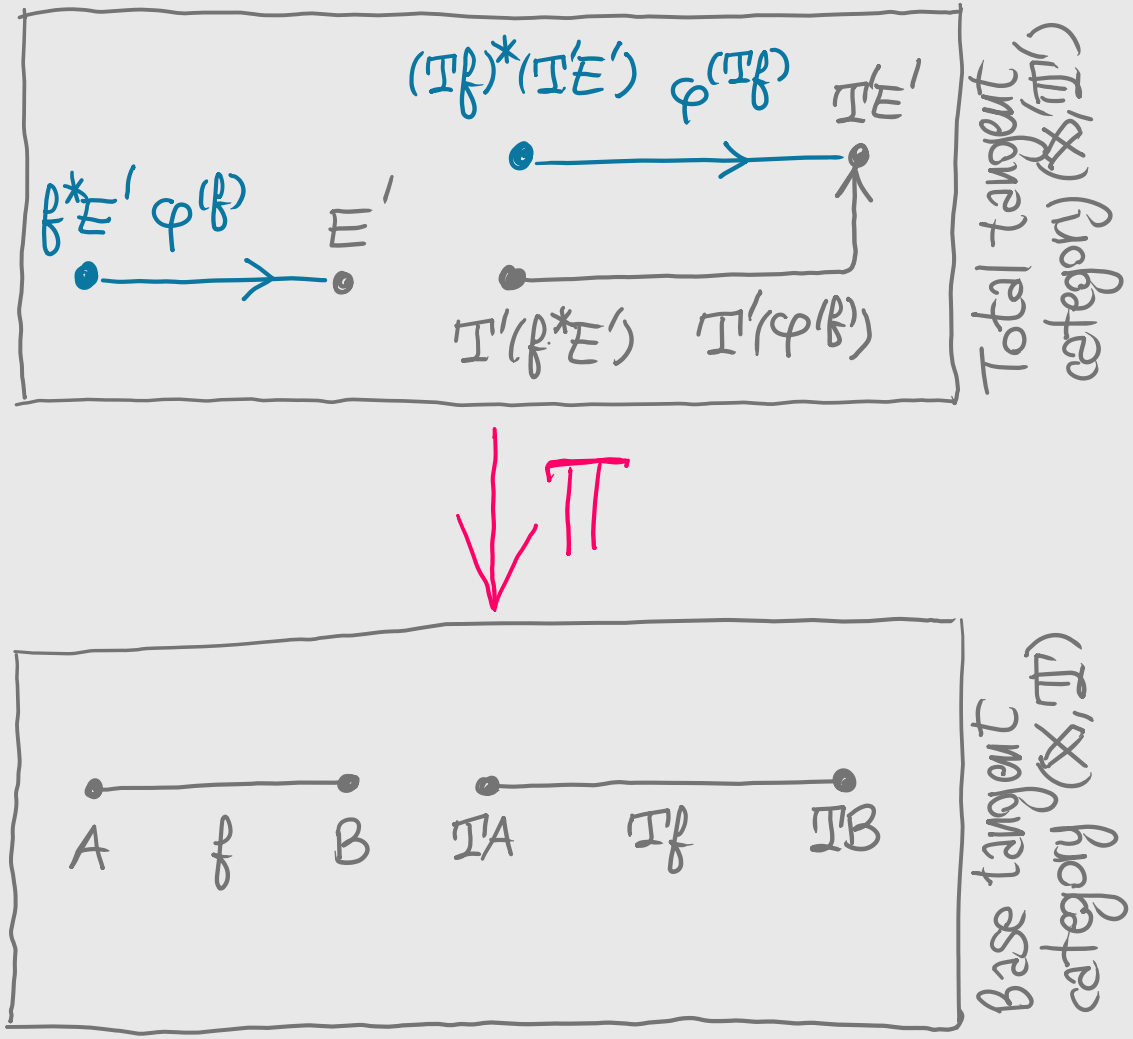

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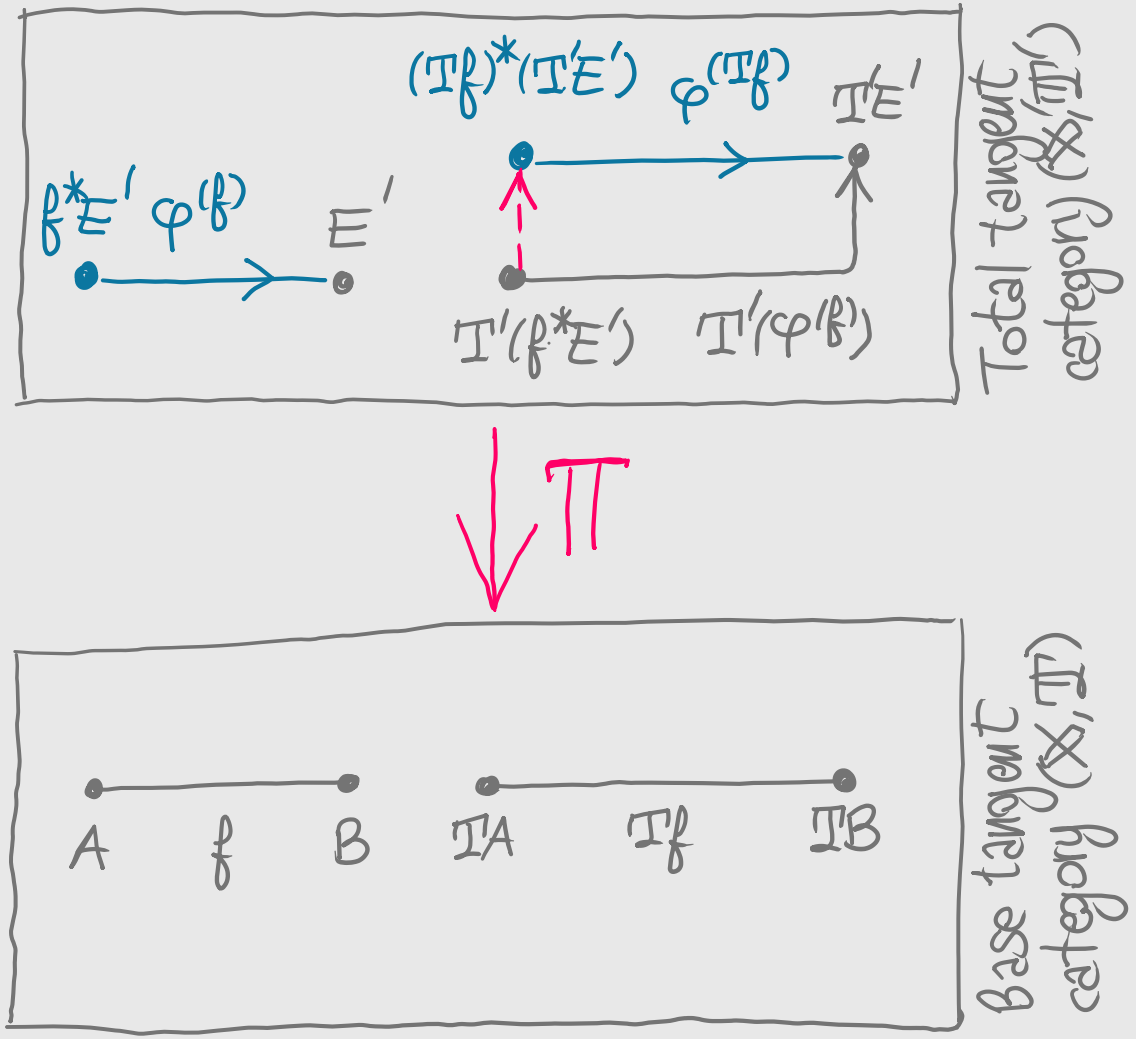

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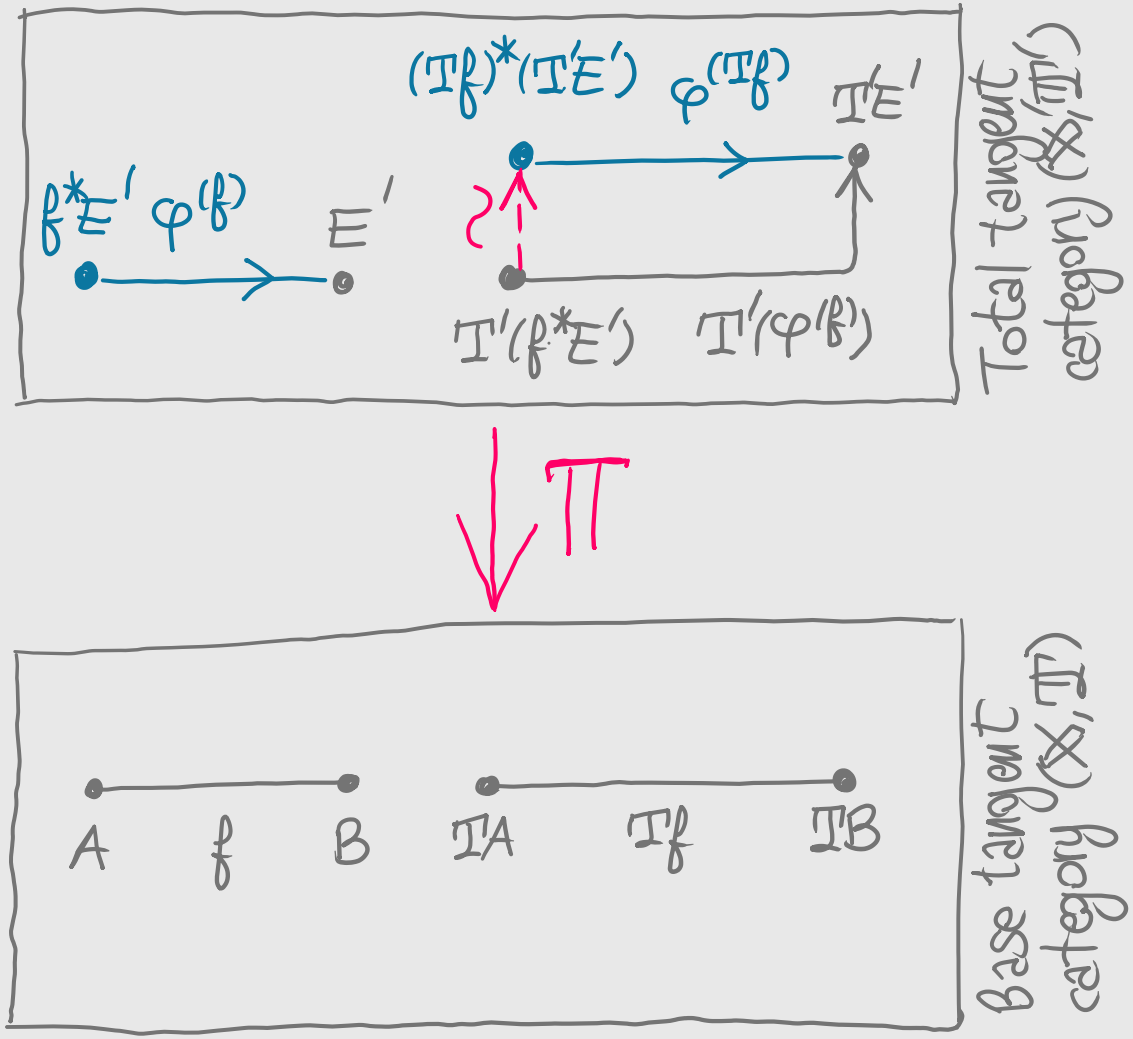

Tangent categories and tangent fibrations

theorem

The fibres of a tangent fibration are tangent categories.

The substitution functors are strong tangent morphisms.

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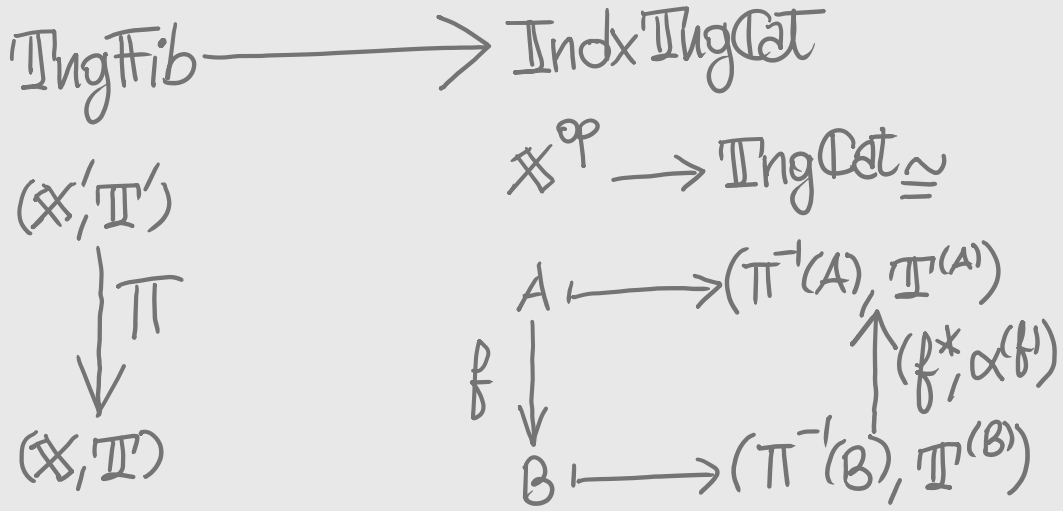

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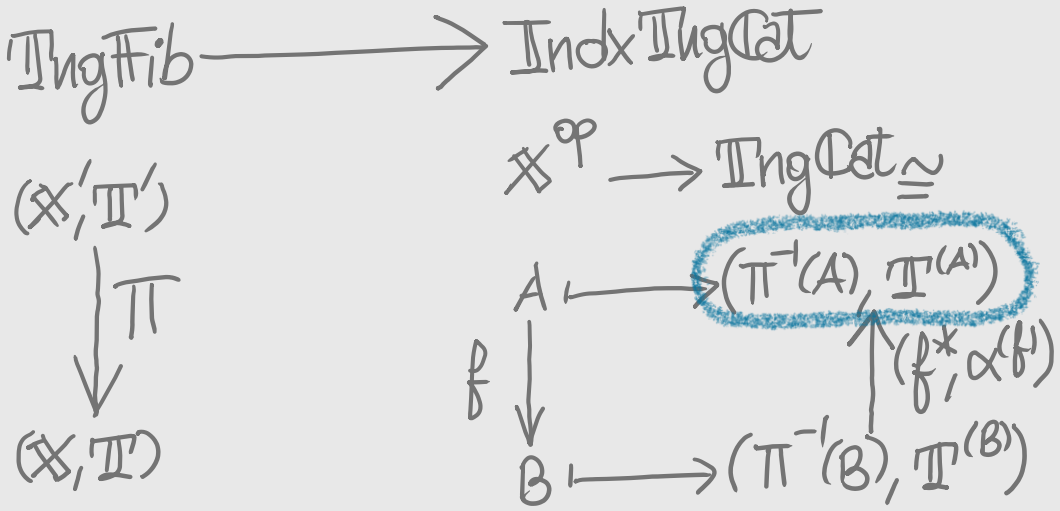

$$\mathbb{T}^{(A)} : \pi^{-1}(A) \xrightarrow{\mathbb{T}'} \pi^{-1}(\mathbb{T}A)$$

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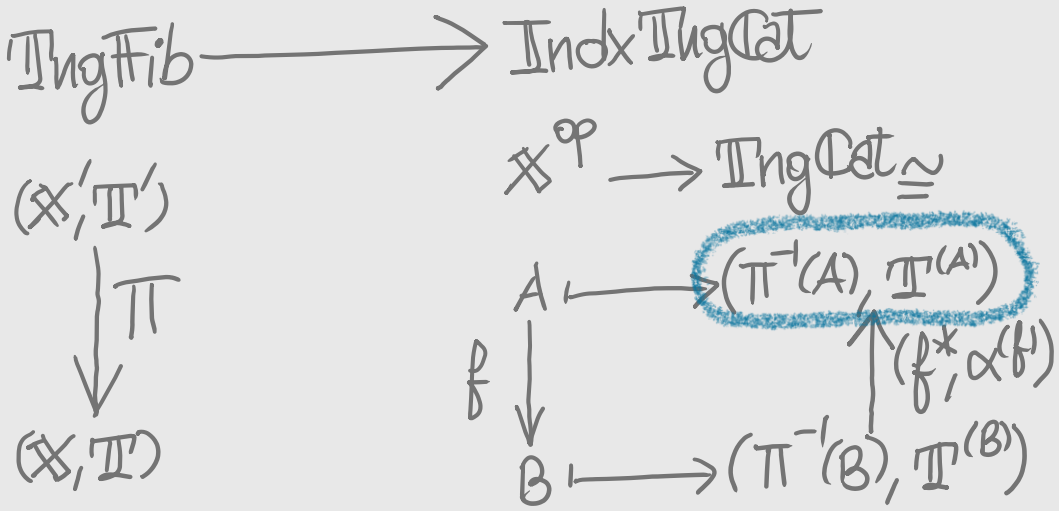


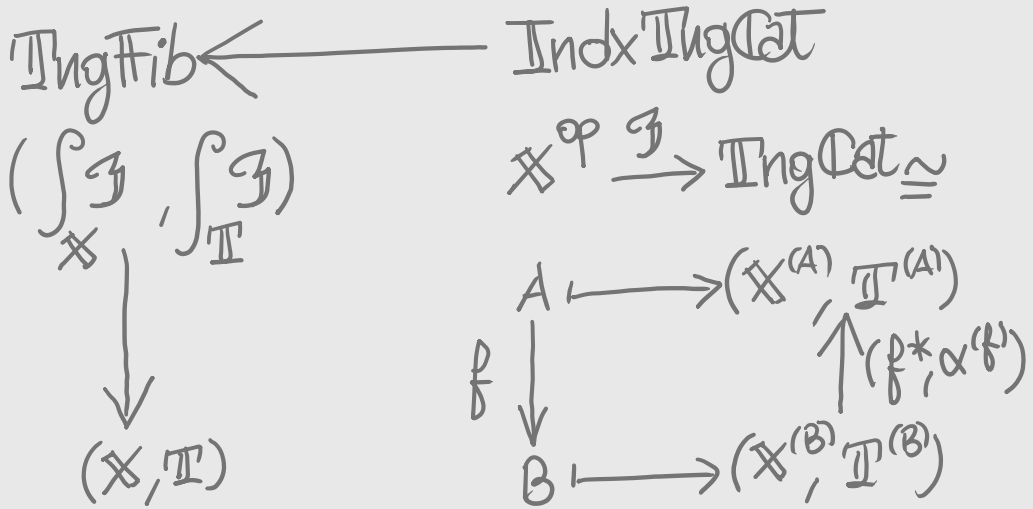
$$\mathbb{T}^{(A)} : \pi^{-1}(A) \xrightarrow{\mathbb{T}'} \pi^{-1}(\mathbb{T}A) \xrightarrow{z^*} \pi^{-1}(A)$$

$$A \xrightarrow{z} \mathbb{T}A$$

Tangent categories and tangent fibrations

theorem

The Grothendieck construction sends an indexed tangent category to a cloven tangent fibration.



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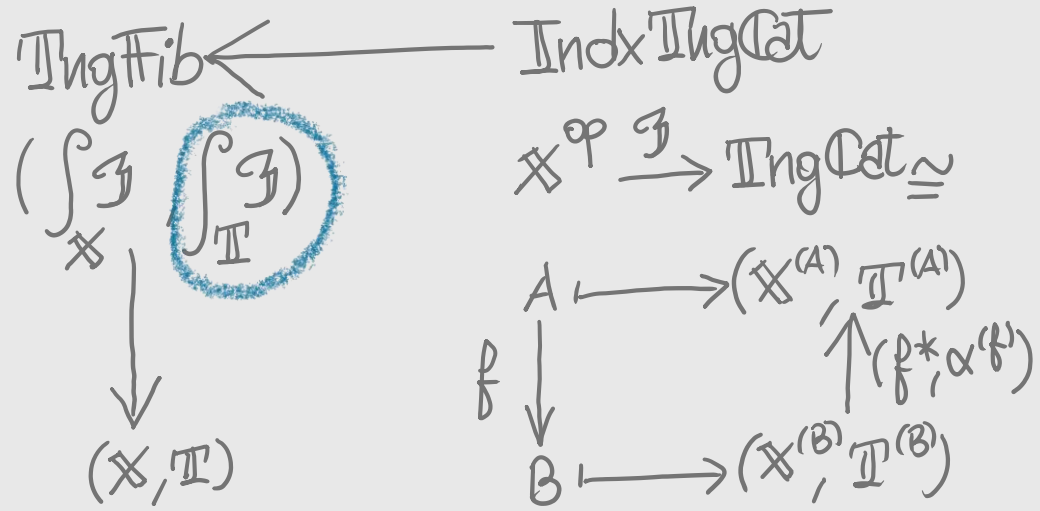

$\mathbb{T}'E := \mathbb{T}^{(A)}E$

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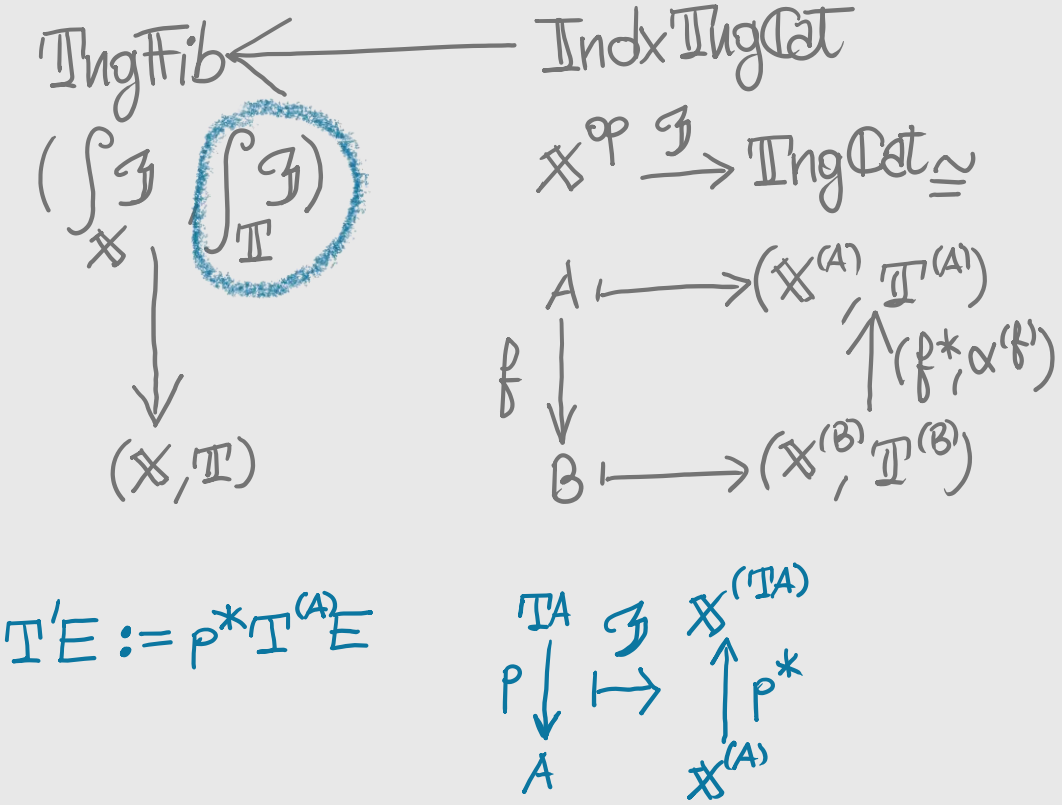

Tangent categories and tangent fibrations

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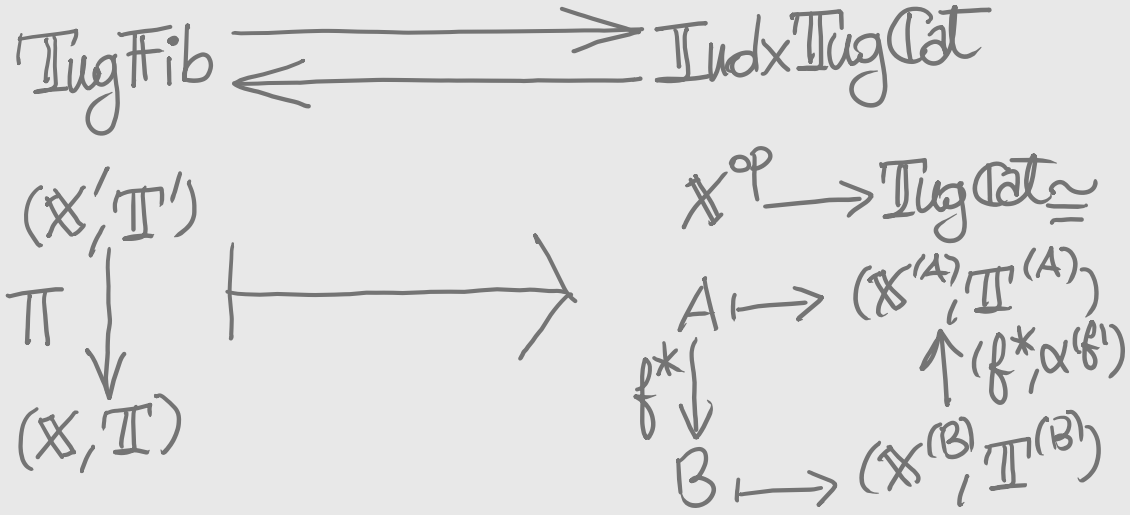
The two 2-functors form an adjunction, but not an equivalence.



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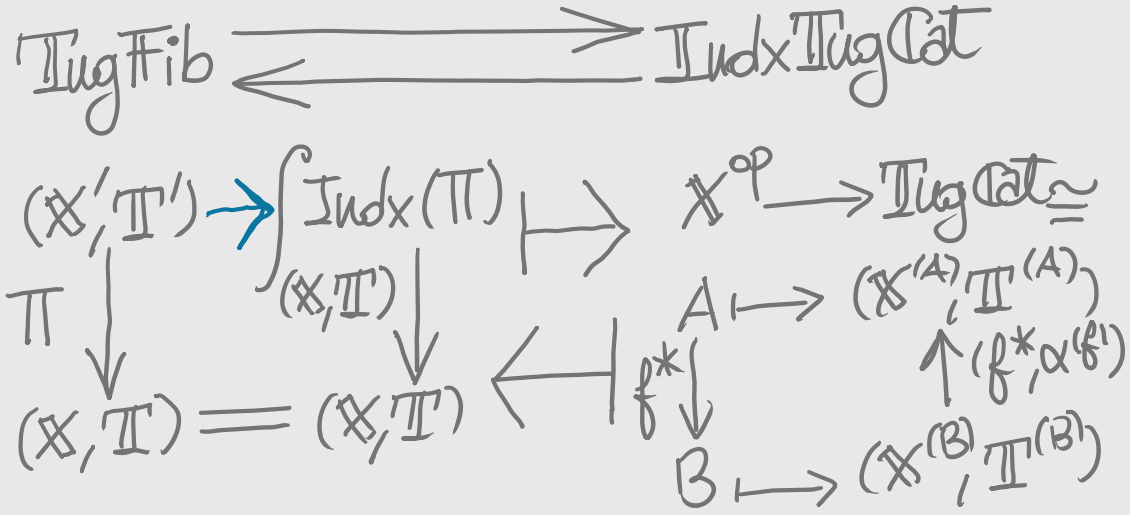

$$\tilde{T}E = p^* T^{(A)} E = p^* x^* T' E$$

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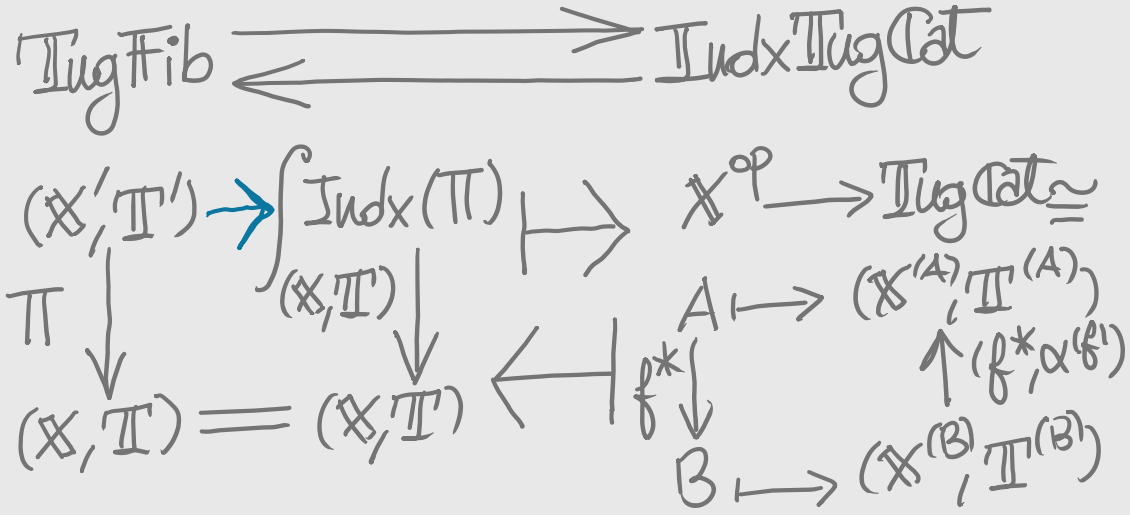

$$\tilde{\mathbb{T}}E = p^*\mathbb{T}^{(A)}E = p^*z^*\mathbb{T}'E \neq \mathbb{T}'E$$

When we take z^* we loose information about the tangent bundle functor

Tangent objects

chapter 2

"A tangent object
is an object of a 2-category
with a tangent structure."

Tangent categories and tangent fibrations

definition

A **tangent object** consists of:

$\mathbb{X} \in \mathbb{C}$

Object in a 2-category
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Object in a 2-category

Tangent bundle morphism

$\mathbb{X} \xrightarrow{\mathbb{T}} \mathbb{X}$

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Tangent categories and tangent fibrations

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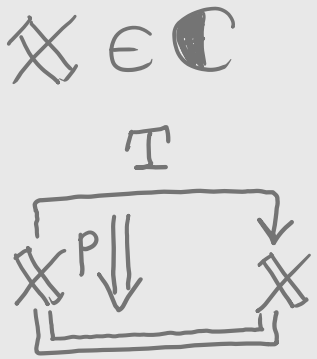

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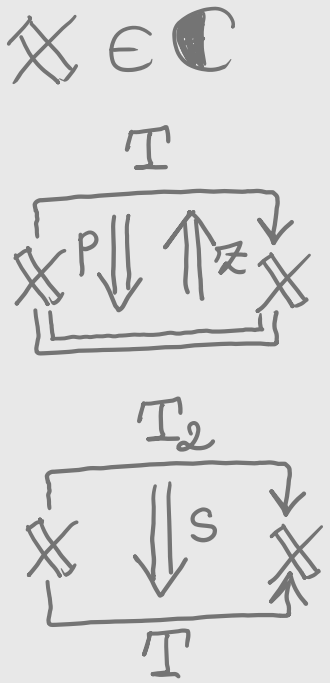

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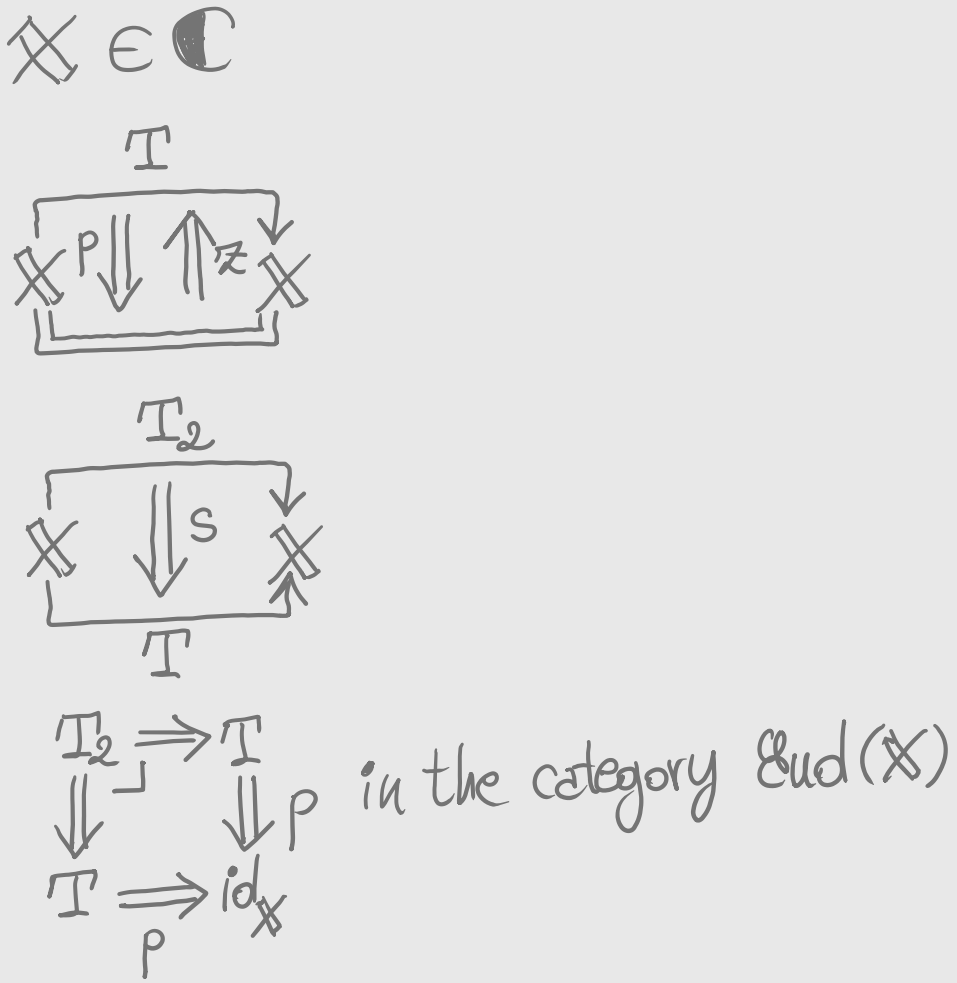

in the category $\mathbb{End}(\mathbb{X})$

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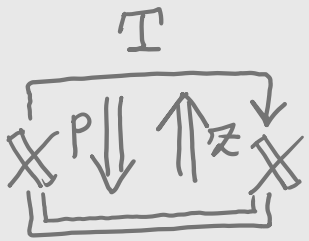

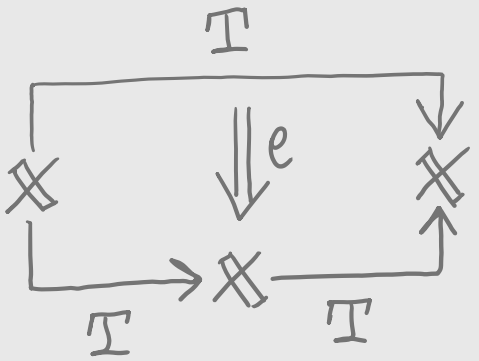

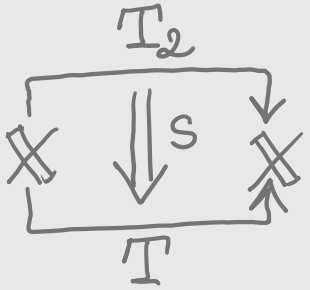

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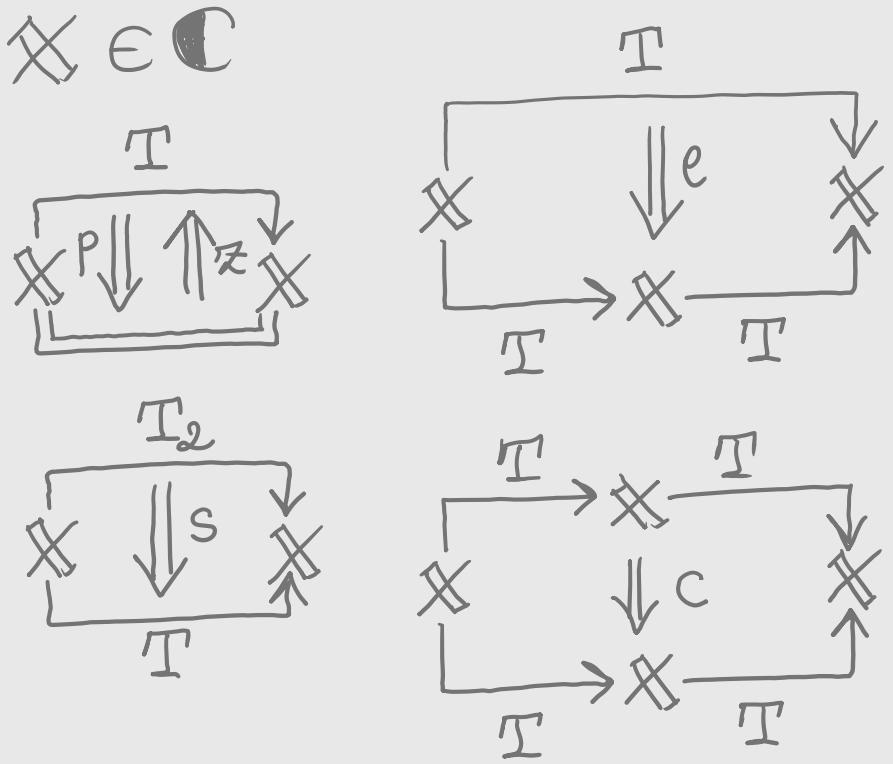

lemma

Tangent objects in **Cat** are tangent categories.

$$\mathbb{T}ug(\mathbb{C}at) \cong \mathbb{T}ug\mathbb{C}at$$

Tangent categories and tangent fibrations

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Tangent objects in **Mnd** are tangent monads.

$$\mathbb{T}ug(\mathbb{C}at) \cong \mathbb{T}ug\mathbb{C}at$$

$$\mathbb{T}ug(\mathbb{M}nd(\mathbb{C})) \cong \mathbb{M}nd(\mathbb{T}ug(\mathbb{C}))$$

Tangent categories and tangent fibrations

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$$\mathbb{T}ug(\mathbb{C}at) \cong \mathbb{T}ug\mathbb{C}at$$

$$\mathbb{T}ug(\mathbb{M}nd(\mathbb{C})) \cong \mathbb{M}nd(\mathbb{T}ug(\mathbb{C}))$$

$$\mathbb{T}ug(\mathbb{M}nd(\mathbb{C}at)) \cong \mathbb{M}nd(\mathbb{T}ug(\mathbb{C}at))$$

$$\cong \mathbb{M}nd(\mathbb{T}ug\mathbb{C}at) \cong \mathbb{T}ug\mathbb{M}nd$$

Tangent categories and tangent fibrations

lemma

Tangent objects in **Cat** are tangent categories.

Tangent objects in **Mnd** are tangent monads.

Tangent objects in **Fib** are tangent fibrations.

$$\mathbb{T}\mathrm{ug}(\mathbb{C}\mathrm{at}) \cong \mathbb{T}\mathrm{ug}\mathbb{C}\mathrm{at}$$

$$\mathbb{T}\mathrm{ug}(\mathrm{Mnd}(\mathbb{C})) \cong \mathrm{Mnd}(\mathbb{T}\mathrm{ug}(\mathbb{C}))$$

$$\mathbb{T}\mathrm{ug}(\mathrm{Mnd}(\mathbb{C}\mathrm{at})) \cong \mathrm{Mnd}(\mathbb{T}\mathrm{ug}(\mathbb{C}\mathrm{at}))$$

$$\cong \mathrm{Mnd}(\mathbb{T}\mathrm{ug}\mathbb{C}\mathrm{at}) \cong \mathbb{T}\mathrm{ug}\mathrm{Mnd}$$

$$\mathbb{T}\mathrm{ug}(\mathbb{F}\mathrm{ib}) \cong \mathbb{T}\mathrm{ug}\mathbb{F}\mathrm{ib}$$

The tangent Grothendieck construction

chapter 3

"Tangent fibrations
are tangent objects in **Fib**.
They are equivalent to
tangent indexed categories."

The tangent Grothendieck construction

lemma

The operation which sends
a 2-category $\mathbf{C}$ to the 2-category
$\mathbf{Tng}(\mathbf{C})$ of tangent objects
of $\mathbf{C}$ extends to a 2-functor.

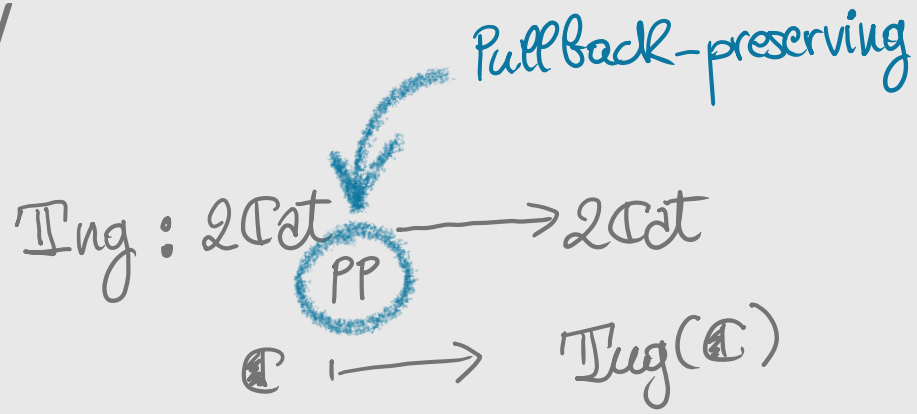
$$\mathbb{T}ng : 2\mathbb{C}at_{PP} \longrightarrow 2\mathbb{C}at$$

$$\mathbb{C} \longmapsto \mathbb{T}ng(\mathbb{C})$$

The tangent Grothendieck construction

lemma

The operation which sends
a 2-category $\mathbf{C}$ to the 2-category
$\mathbf{Tng}(\mathbf{C})$ of tangent objects
of $\mathbf{C}$ extends to a 2-functor.



The tangent Grothendieck construction

Lanfranchi 2023

theorem

Tangent fibrations are equivalent
to tangent indexed categories,
i.e. tangent objects in
the 2-category **IndxCat**.

The tangent Grothendieck construction

theorem Tangent fibrations are equivalent to tangent indexed categories, i.e. tangent objects in the 2-category **IndxCat**.

$$\begin{array}{c} \mathbb{F}\mathrm{ib} \cong \mathbb{I}\mathrm{ndx}\mathbb{C}\mathrm{at} \\ \downarrow \\ \mathbb{T}\mathrm{ang}\mathbb{F}\mathrm{ib} \cong \mathbb{T}\mathrm{ang}(\mathbb{F}\mathrm{ib}) \cong \mathbb{T}\mathrm{ang}(\mathbb{I}\mathrm{ndx}\mathbb{C}\mathrm{at}) \end{array}$$

definition

A **tangent indexed category** consists of:

Base tangent category $(\mathbb{X}, \mathbb{T})$

Pseudofunctor

Indexed tangent functor

Natural transformations

definition

A **tangent indexed category** consists of:

Base tangent category $(\mathbb{X}, \mathbb{T})$

Pseudofunctor $\mathbb{X}^{op} \longrightarrow \mathbb{C}at$

$A \mapsto \mathbb{X}^{(A)}$

$f \mapsto f^*$

$B \mapsto \mathbb{X}^{(B)}$

Indexed tangent functor

Natural transformations

definition

A **tangent indexed category** consists of:

Base tangent category
Pseudofunctor
Indexed tangent functor
Natural transformations

$(\mathbb{X}, \mathbb{T})$

$$\mathbb{X}^{op} \longrightarrow \mathbb{C}at$$

$$A \mapsto \mathbb{X}^{(A)}$$

$$f \downarrow \mapsto \uparrow f^*$$

$$B \mapsto \mathbb{X}^{(B)}$$

$$\mathbb{T}^{(A)} : \mathbb{X}^{(A)} \longrightarrow \mathbb{X}^{(\mathbb{T}A)}$$

the end.

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https://arxiv.org/abs/2311.14643