Dalhousie University
Mathematics & Statistics

The differential bundles of $\mathscr{P}$-affine schemes

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Supervisors: Dorette Pronk Geoff Cruttwell

Chapter 0

The plan for today

"What does the geometric tangent structure tell us about affine schemes over an operad?"

The plan for today

Differential bundles

The plan for today

Differential bundles

Slice tangent category

The plan for today

Differential bundles

Slice tangent category

Functoriality

The plan for today

Differential bundles

Slice tangent category

Functoriality

Enveloping operad

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Enveloping operad

The classification

Differential bundles

Cockett, Cruttwell 2017

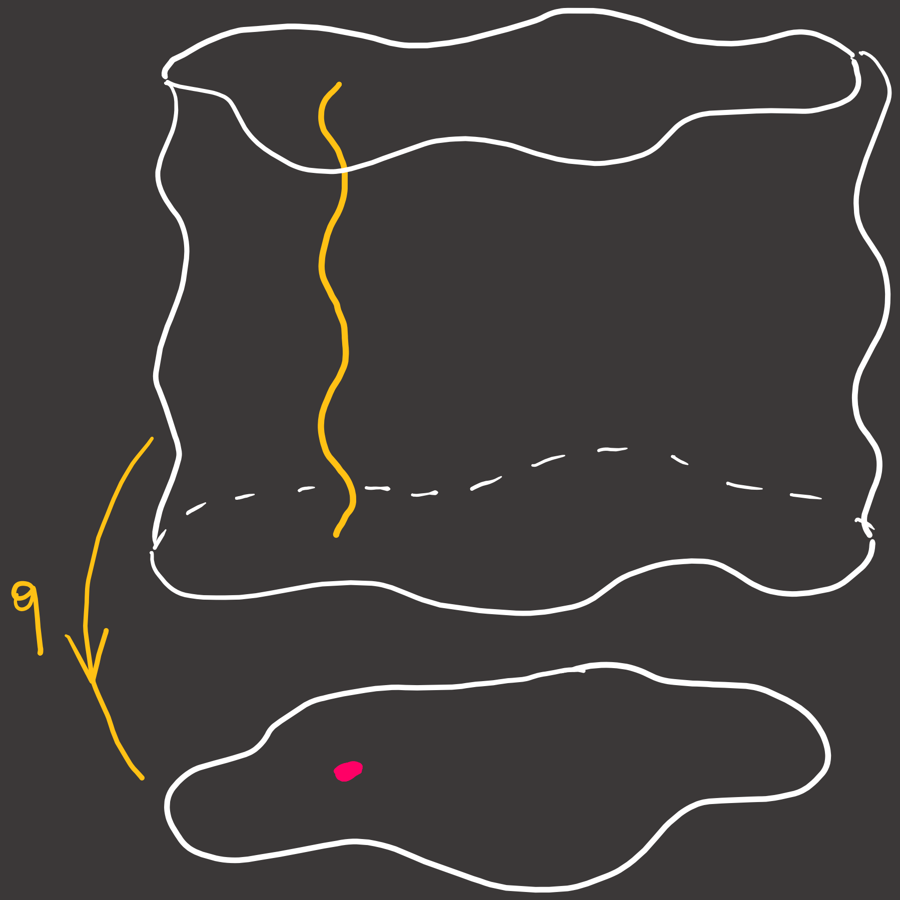
Chapter 1

"Differential bundles
are vector bundles
in a tangent category."

Differential bundles

Projection

$$E \xrightarrow{q} M$$



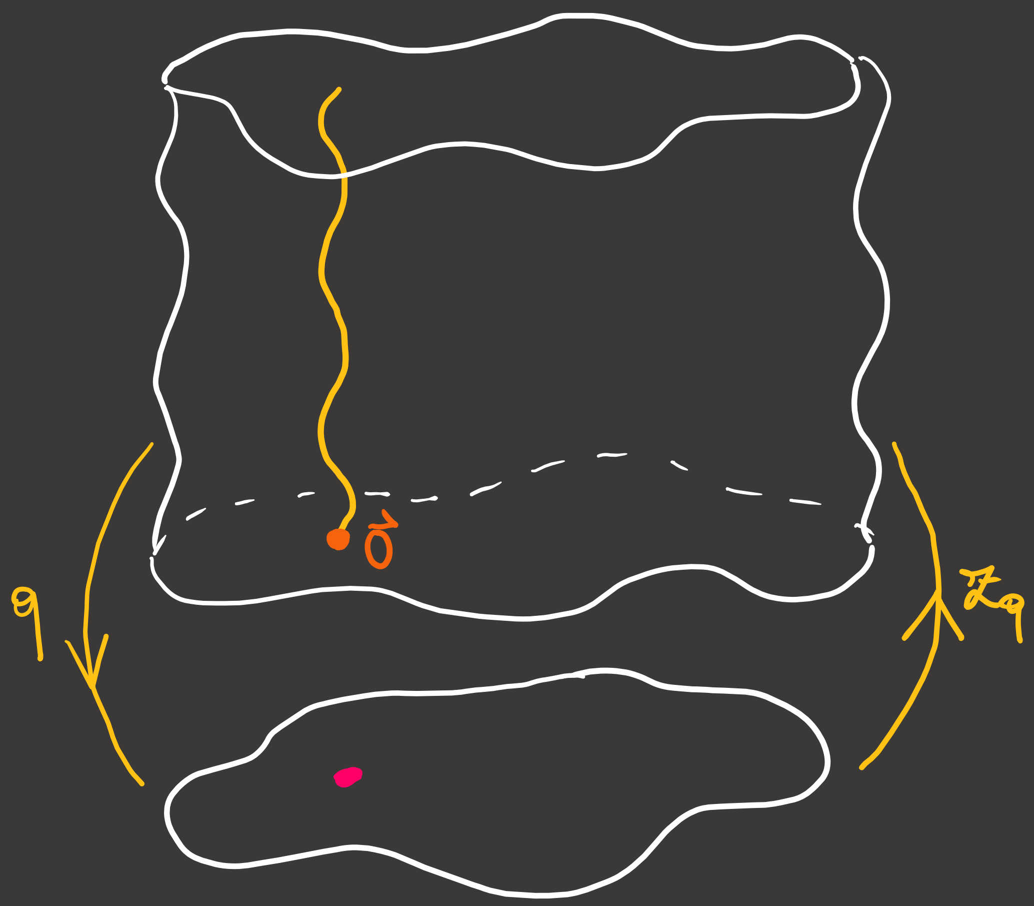
Differential bundles

Cockett, Cruttwell 2017

Projection

Zero morphism

$$E \underset{z_q}{\overset{q}{\rightleftarrows}} M$$



Differential bundles

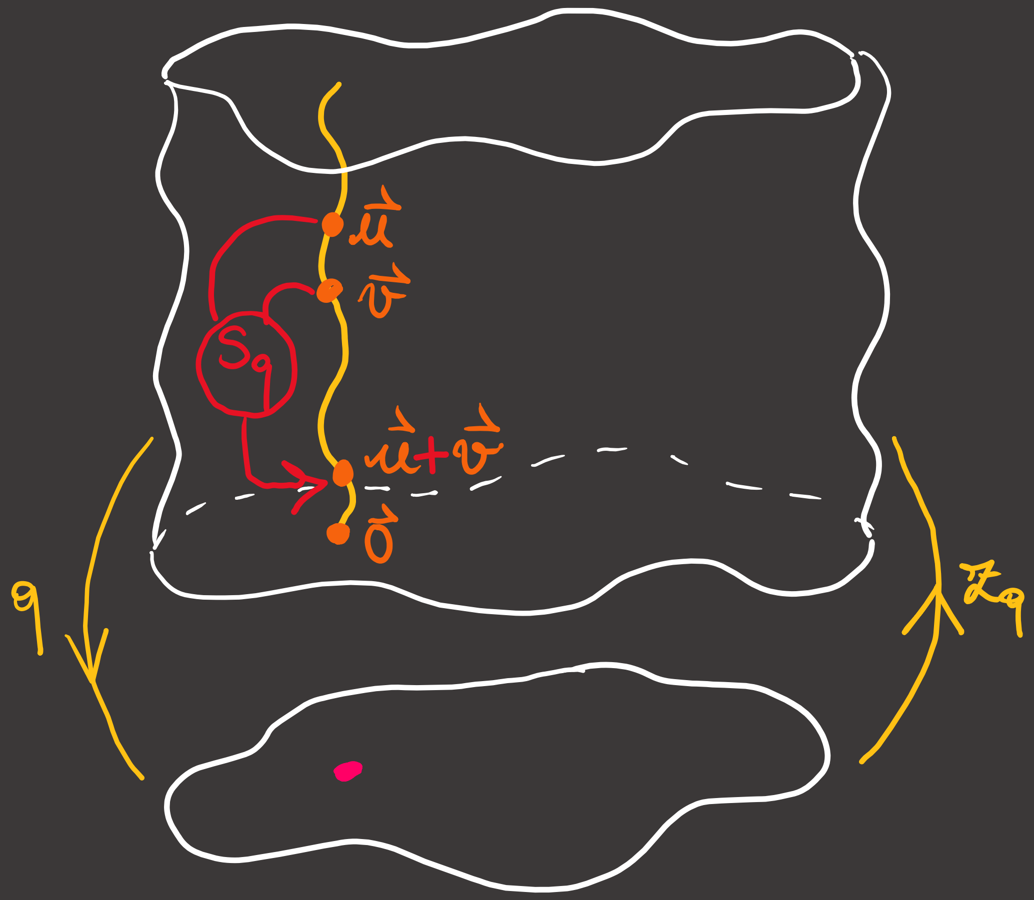
Cockett, Cruttwell 2017

Projection

Zero morphism

Sum morphism

$$\begin{tikzcd} E_2 \arrow[r, "s_q"] & E \arrow[r, shift left, "q"] & M \arrow[l, shift left, "z_q"] \end{tikzcd}$$



Differential bundles

Cockett, Cruttwell 2017

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$$\begin{tikzcd} E_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & M \end{tikzcd}$$

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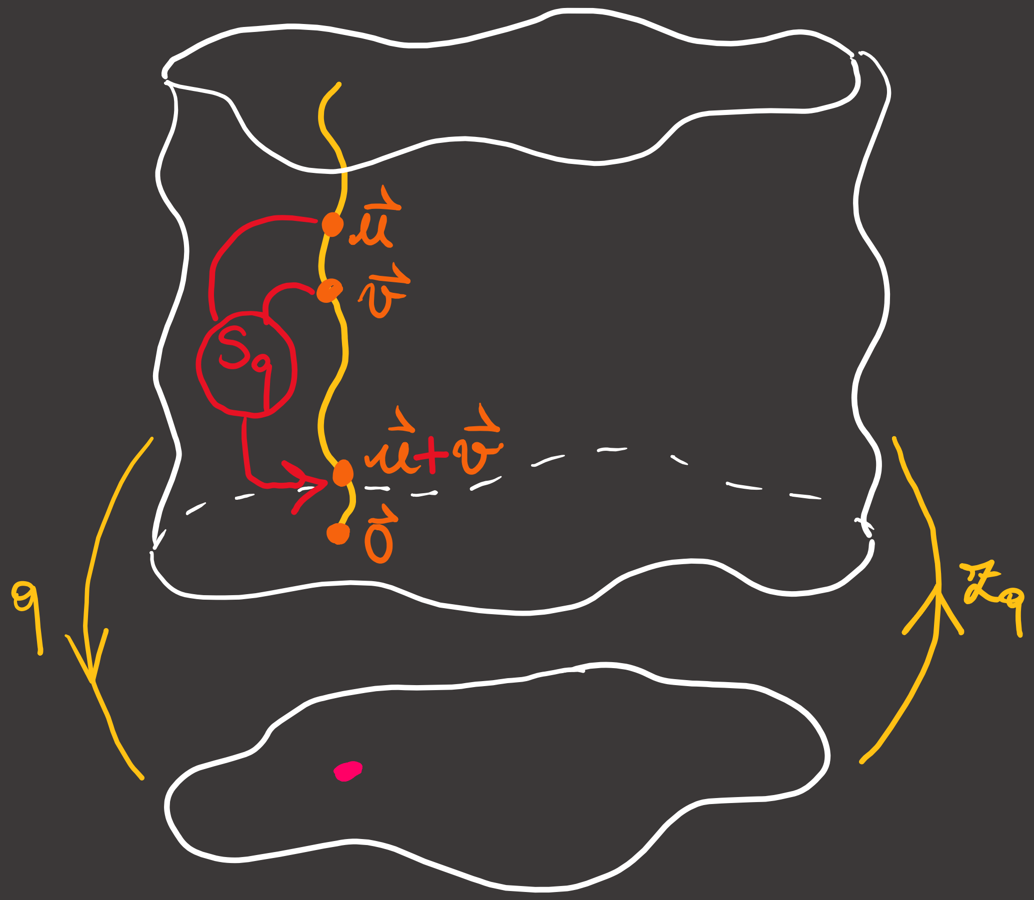

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$\ell_q : E \longrightarrow TE$

Differential bundles

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$(x; u) \longmapsto (x, 0, 0, u)$

Differential bundles

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$$\begin{tikzcd}
E_2 \arrow[r, "\mu_q"] \arrow[d, "\pi_1 q"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tq"] \\
M \arrow[r, "\zeta"'] & TM
\end{tikzcd}$$

Differential bundles

Cockett, Cruttwell 2017

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$\mu_q := (\ell_q \times \mathcal{Z}) T s_q$

Theorem

Differential bundles
in the tangent category of
smooth manifolds are
equivalent to vector bundles*.

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*vector bundles defined as bundles
whose fibres are vector spaces
not necessarily isomorphic
to a typical fibre.

Differential bundles

Cruttwell, Lemay 2023

Theorem

Differential bundles
in the tangent category
of (commutative) affine schemes
are modules over algebras.

Cruttwell, Lemay 2023

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$(cAlg^{op}, T^{o})$

$T^{o}A := Sym_A \Omega A$

Cruttwell, Lemay 2023

Theorem

Differential bundles
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this is the geometric tangent category of the operad Com

Theorem

Differential bundles in the tangent category of (commutative) affine schemes are modules over algebras.

$(\mathsf{cAlg}^{op}, \mathbb{T}^{o})$

$\mathbb{T}^{o}A := \mathrm{Sym}_A\,\Omega A$

We want to extend this result to any operads

this is the geometric tangent category of the operad Com

Differential bundles

Cockett, Cruttwell 2017

Differential bundles over the terminal object are called differential objects.

Product diagram

$$A \xleftarrow{p} TA \xrightarrow{\hat{p}} A$$

Differential bundles

Cockett, Cruttwell 2017

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Differential bundles

Cockett, Cruttwell 2017

Differential bundles over the terminal object are called differential objects.

In differential geometry:

$\mathbb{R}^n$, $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

Product diagram

Differential projection

$$A \xleftarrow{p} TA \xrightarrow{\hat{p}} A$$

$$u \longleftarrow\!\shortmid (u, u) \shortmid\!\longrightarrow u$$

The slice tangent category

Rosický 1984
Cockett, Cruttwell 2017

Chapter 2

"The slice of a tangent category is again a tangent category."

The slice tangent category

Rosický 1984
Cockett, Cruttwell 2017

Slice category over
sliceable objects

The slice tangent category

Rosický 1984
Cockett, Cruttwell 2017

Slice category

$q : E \longrightarrow M$

$$\begin{tikzcd}
T^M E \arrow[r, "\iota^M"] \arrow[d, "q^M"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tq"] \\
M \arrow[r, "z"'] & TM
\end{tikzcd}$$

The slice tangent category

Rosický 1984
Cockett, Cruttwell 2017

Slice category

$$T^M : \mathbb{X}/M \longrightarrow \mathbb{X}/M$$

Tangent bundle functor

$$q : E \longrightarrow M$$

$$\begin{tikzcd}
T^M E \arrow[r, "\iota^M"] \arrow[d, "q^M"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tq"] \\
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The slice tangent category

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Cockett, Cruttwell 2017

Slice category

$$\mathrm{T}^M : \mathbb{X}/M \longrightarrow \mathbb{X}/M$$

Tangent bundle functor

$$(q : E \to M) \longmapsto (\mathrm{T}^M E \xrightarrow{q^M} M)$$

$$q : E \longrightarrow M$$

$$\begin{tikzcd}
\mathrm{T}^M E \arrow[r, "\iota^M"] \arrow[d, "q^M"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}$$

The slice tangent category

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Tangent bundle functor

$$q : E \to M$$

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$$T^M : \mathbb{X}/M \longrightarrow \mathbb{X}/M$$

$$(q : E \to M) \longmapsto (T^M E \xrightarrow{q^M} M)$$

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "q"'] & & E' \arrow[dl, "q'"] \\
& M &
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
& T^M E' \arrow[rr, "\iota^M"] \arrow[dd, "q'^M" near start] & & TE' \arrow[dd, "Tq'"] \\
T^M E \arrow[ur, dashed, "T^M f"] \arrow[rr, "\iota^M" near start] \arrow[dd, "q^M"'] & & TE \arrow[ur, "Tf"] \arrow[dd, "Tq" near start] & \\
& M \arrow[rr, "z" near start] & & TM \\
M \arrow[ur, equal] \arrow[rr, "z"'] & & TM \arrow[ur, equal] &
\end{tikzcd}$$

The slice tangent category

Cockett, Cruttwell 2017

Theorem

Differential bundles
are differential objects
in the slice tangent category.

The slice tangent category

Cockett, Cruttwell 2017

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Universality of ℓ_q:

$$
\begin{tikzcd}
E_2 \arrow[r, "u_q"] \arrow[d, "\pi_1 q"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tq"] \\
M \arrow[r, "\zeta"'] & TM
\end{tikzcd}
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The slice tangent category

Cockett, Cruttwell 2017

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\end{tikzcd}$$

this is T^M_q is the slice tangent category!

The slice tangent category

Cockett, Cruttwell 2017

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Universality of e_q:

$$\begin{tikzcd} E_2 \arrow[r, "u_q"] \arrow[d, "\pi_1 q"'] \arrow[dr, phantom, "\lrcorner" very near start] & TE \arrow[d, "Tq"] \\ M \arrow[r, "\zeta"'] & TM \end{tikzcd}$$

this is T^M_q is the slice tangent category!

But $E_2 \longrightarrow M = \begin{matrix} E \\ q\downarrow \\ M \end{matrix} \times_M \begin{matrix} E \\ \downarrow \\ M \end{matrix}$

The slice tangent category

Cockett, Cruttwell 2017

Theorem

Differential bundles
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Universality of ℓ_q:

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M \arrow[r, "\zeta"'] & TM
\end{tikzcd}$$

this is T^M_q is the slice tangent category!

But $E_2 \longrightarrow M = q\downarrow \times_M \downarrow$ with $E \xrightarrow{q} M$ and $E \to M$, i.e. $\begin{matrix} E \\ {\scriptstyle q}\downarrow \\ M \end{matrix} \times_M \begin{matrix} E \\ \downarrow \\ M \end{matrix}$

this is a product in $\mathbb{X}/M$

The slice tangent category

A new characterization of the slice tangent category:

$$\mathrm{Term} : \mathsf{tTngCat} \longrightarrow \mathsf{TngPair} \qquad \mathrm{Slice} : \mathsf{TngPair} \longrightarrow \mathsf{tTngCat}$$

$$(\mathbb{X}, \mathbb{T}) \longmapsto (\mathbb{X}, \mathbb{T}; \mathbb{1}) \qquad (\mathbb{X}, \mathbb{T}; M) \longmapsto (\mathbb{X}, \mathbb{T})/M$$

tangent categories with terminal object

$\mathrm{T}\mathbb{1} \xrightarrow{\sim} \mathbb{1}$

tangent pairs:

$(\mathbb{X}, \mathbb{T}; M)$

$M \in \mathbb{X}$

Theorem

Slice is a right adjoint of Term.

The slice tangent category

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Theorem

Slice is a right adjoint of Term.

Let $(F,a;f)\colon(X,T;M)\rightarrow(X',T';M')$ be a tangent morphism.

If the following diagrams are pullbacks:

$$
\begin{tikzcd}
FTE \arrow[r, "\alpha"] \arrow[d, "FTq"'] & T'FE \arrow[d, "T'Fq"] \\
FTM \arrow[r, "\alpha"'] & T'FM
\end{tikzcd}
\qquad
\begin{tikzcd}
FM \arrow[r, "z'F"] \arrow[d, "\varphi"'] & T'FM \arrow[d, "T'\varphi"] \\
M \arrow[r, "z'"'] & T'M
\end{tikzcd}
\qquad
\begin{tikzcd}
FT^{(M)}E \arrow[r, "F\iota^{(M)}"] \arrow[d, "Fq^*"'] & FTE \arrow[d, "FTq"] \\
FM \arrow[r, "Fz"'] & FTM
\end{tikzcd}
$$

Slice(F,a) is a strong tangent morphism.

Chapter 3

"The operation which associates an operad with the geometric tangent category is functiorial."

Functoriality

Morphisms of operads

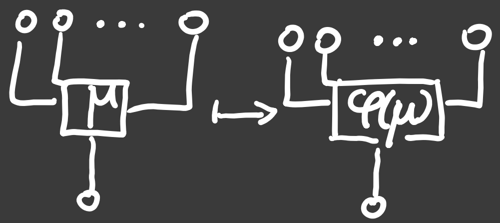
$\mathcal{P} \longrightarrow \mathcal{Q}$

Functoriality

Morphisms of operads

$\mathcal{P} \longrightarrow \mathcal{Q}$

$\mathcal{P}(n) \xrightarrow{\varphi_n} \mathcal{Q}(n)$



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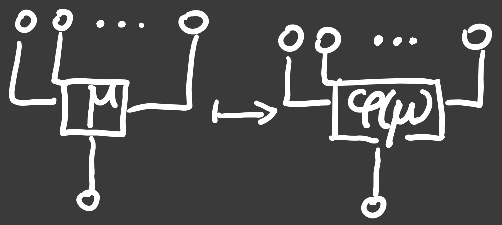

$\mathrm{Operad} \longrightarrow \mathrm{InfMod}(\mathrm{Mod}_R, \mathbb{I})$

$\mathcal{P} \longmapsto (S_{\mathcal{P}}, \partial)$

Functoriality

Morphisms of operads

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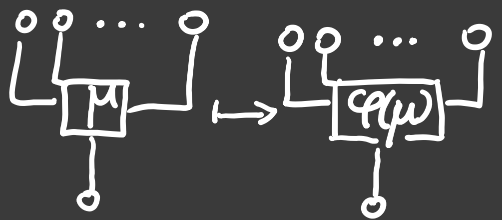

$$\mathrm{Operad} \longrightarrow \mathrm{IngMod}(\mathrm{Mod}_R, \mathbb{T})$$

$$\mathcal{P} \longmapsto (S_{\mathcal{P}}, \partial)$$

Biproducts

Functoriality

Lanfranchi 2023

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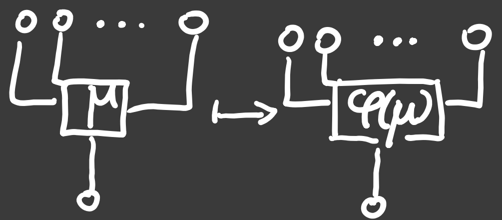

$\mathbf{Operad} \longrightarrow \mathbb{T}\mathrm{ng}\mathbf{Mnd}(\mathrm{Mod}_R, \mathbb{T})$

$\mathcal{P} \longmapsto (S_{\mathcal{P}}, \partial)$

Biproducts

$\mathbb{T}\mathrm{ng}\mathbf{Mnd}(\mathbb{X}, \mathbb{T})^{op} \longrightarrow \mathbb{T}\mathrm{ng}\mathbf{Cat}$

$(S, \alpha) \longmapsto (\mathrm{Alg}(S), \mathbb{T}(S))$

Functoriality

Lanfranchi 2023

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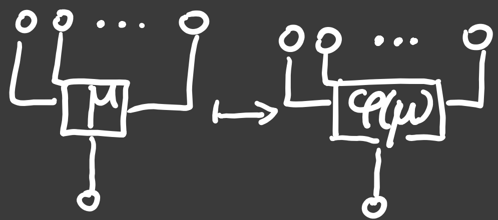

$\mathbf{Operad} \longrightarrow \mathbf{TngMnd}(\mathrm{Mod}_R, \mathbb{T})$

$\mathcal{P} \longmapsto (S_{\mathcal{P}}, \partial)$

Biproducts

$\mathbf{TngMnd}(\mathbb{X}, \mathbb{T})^{op} \longrightarrow \mathbf{TngCat}$

$(S, \alpha) \longmapsto (\mathrm{Alg}(S), \mathbb{T}(S))$

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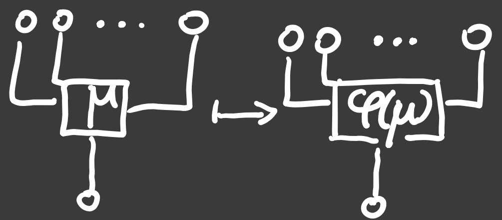

$$\mathrm{Operad} \longrightarrow \mathrm{TngMnd}(\mathrm{Mod}_R, \mathbb{T})$$

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$$(S, \alpha) \longmapsto (\mathrm{Alg}(S), \mathbb{T}(S))$$

$$\mathrm{Operad}^{op} \longrightarrow \mathrm{TngCat}$$

$$\mathcal{P} \longmapsto \mathrm{Alg}(\mathcal{P}) := (\mathrm{Alg}_{\mathcal{P}}, \mathbb{T}_{\mathcal{P}})$$

$$(\varphi : \mathcal{P} \to \mathcal{Q}) \mapsto (\varphi^*, \alpha^*) : \mathrm{Alg}(\mathcal{Q}) \to \mathrm{Alg}(\mathcal{P})$$

Functoriality

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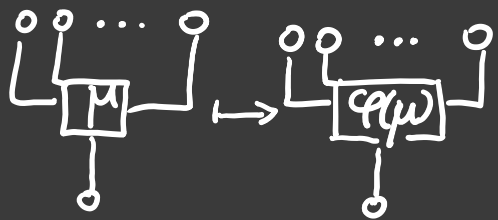

$\mathbf{Operad} \longrightarrow \mathbf{TngMnd}(\mathrm{Mod}_R, \mathbb{T})$

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$$(\varphi : \mathcal{P} \to \mathcal{Q}) \longmapsto (\varphi^*, \alpha^*) : \mathrm{Alg}(\mathcal{Q}) \to \mathrm{Alg}(\mathcal{P})$$

$$\varphi^*(A, \mathcal{Q}(n) \otimes A^{\otimes n} \xrightarrow{\theta} A) =$$
$$= (A, \mathcal{P}(n) \otimes A^{\otimes n} \xrightarrow{\varphi \otimes A^{\otimes n}} \mathcal{Q}(n) \otimes A^{\otimes n} \xrightarrow{\theta} A)$$

Functoriality

Lanfranchi 2023

Morphisms of operads

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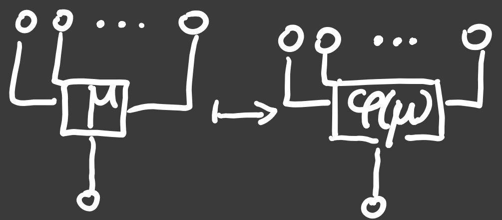

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$\varphi^* T_{\mathcal{Q}} = T_{\mathcal{P}} \varphi^*$

$$\varphi^*(A, \mathcal{Q}(n) \otimes A^{\otimes n} \xrightarrow{\theta} A) =$$

$$= (A, \mathcal{P}(n) \otimes A^{\otimes n} \xrightarrow{\varphi \otimes A^{\otimes n}} \mathcal{Q}(n) \otimes A^{\otimes n} \xrightarrow{\theta} A)$$

Functoriality

$\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$

$$\mathbb{A}lg_{\mathcal{P}} \xleftarrow[\varphi^*]{} \mathbb{A}lg_{\mathcal{Q}}$$

Functoriality

$$\frac{\varphi : \mathcal{P} \to \mathcal{Q}}{\begin{tikzcd} \mathbb{A}\mathrm{lg}_{\mathcal{P}} \arrow[r, shift left, "\varphi_!"] & \mathbb{A}\mathrm{lg}_{\mathcal{Q}} \arrow[l, shift left, "\varphi^*"] \end{tikzcd}}$$

$$\frac{\varphi : \mathcal{P} \to \mathcal{Q}}{\mathrm{Alg}_{\mathcal{P}} \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathrm{Alg}_{\mathcal{Q}}}$$

The double category of tangent cats:

$$\begin{tikzcd}
(\mathbb{X}_\circ, \mathbb{T}_\circ) \arrow[r, "{(F_\circ, \alpha_\circ)}"] \arrow[d, "|" marking, "{(G,\beta)}"'] \arrow[dr, phantom, "\theta\ \Swarrow"] & (\mathbb{X}'_\circ, \mathbb{T}'_\circ) \arrow[d, "|" marking, "{(G',\beta')}"] \\
(\mathbb{X}_\bullet, \mathbb{T}_\bullet) \arrow[r, "{(F_\bullet, \alpha_\bullet)}"'] & (\mathbb{X}'_\bullet, \mathbb{T}'_\bullet)
\end{tikzcd}$$

$\alpha_\circ : F_\circ T_\circ \Rightarrow T'_\circ F_\circ$

$\alpha_\bullet : F_\bullet T_\bullet \Rightarrow T'_\bullet F_\bullet$

Functoriality

$$\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$$

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The double category of tangent cats:

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$\alpha_\circ : F_\circ T_\circ \Rightarrow T'_\circ F_\circ$

$\alpha_\bullet : F_\bullet T_\bullet \Rightarrow T'_\bullet F_\bullet$

$\beta : T_\bullet G \Rightarrow G T_\circ$

$\beta' : T'_\bullet G' \Rightarrow G' T'_\circ$

Functoriality

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$\alpha_\bullet : F_\bullet T_\bullet \Rightarrow T'_\bullet F_\bullet$

$\beta : T_\bullet G \Rightarrow G T_\circ$

$\beta : T'_\bullet G' \Rightarrow G' T'_\circ$

$\theta : F_\bullet G \Rightarrow G' F_\circ$

Lemma

If the underlying functor G
of a colax tangent morphism (G,b)
has a right adjoint F, then
(F,a) is a lax tangent morphism,
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In particular (F,a) and (G,b)
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categories and every conjunction
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Functoriality

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Lemma

If the underlying functor G of a colax tangent morphism (G,b) has a right adjoint F, then (F,a) is a lax tangent morphism, where a is the mate of b.

In particular (F,a) and (G,b) form a conjunction in the double category of tangent categories and every conjunction is of this form.

$$(\mathbb{X},\mathbb{T}) \underset{(G,\beta)}{\overset{F}{\rightleftarrows}} (\mathbb{X}',\mathbb{T}')$$

$$\beta : \mathbb{T}G \Rightarrow G\mathbb{T}'$$

Functoriality

Lanfranchi 2023

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$$(\mathbb{X},\mathbb{T}) \underset{(G,\beta)}{\overset{(F,\alpha)}{\rightleftarrows}} (\mathbb{X}',\mathbb{T}')$$

$$\beta : \mathbb{T}G \Rightarrow G\mathbb{T}'$$

$$\alpha : F\mathbb{T} \xrightarrow{F\mathbb{T}\eta} F\mathbb{T}GF \xrightarrow{F\beta F} FG\mathbb{T}'F \xrightarrow{\varepsilon\mathbb{T}'F} \mathbb{T}'F$$

Functoriality

Lanfranchi 2023

Prop'n

The operation which sends an operad to its algebraic tangent category extends to two functors:

colax and lax!

$$\alpha^*: \varphi^* T_{\mathcal{Q}} = T_{\mathcal{P}} \varphi^*$$

$$\mathrm{Alg}^*: \mathrm{Operad}^{op} \longrightarrow \mathrm{TngCat}$$

$$\mathcal{P} \longmapsto \mathrm{Alg}(\mathcal{P})$$

$$\varphi: \mathcal{P} \to \mathcal{Q} \longmapsto (\varphi^*, \alpha^*): \mathrm{Alg}(\mathcal{Q}) \longrightarrow \mathrm{Alg}(\mathcal{P})$$

Functoriality

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$$\mathscr{P} \longmapsto \mathrm{Alg}(\mathscr{P})$$

NOT an isomorphism

$$\varphi: \mathscr{P} \to \mathscr{Q} \longmapsto (\varphi_!, \alpha_!): \mathrm{Alg}(\mathscr{P}) \longrightarrow \mathrm{Alg}(\mathscr{Q})$$

Lemma

Consider a lax tangent morphism between two adjunctable tangent categories. Then the opposite of F is also a lax tangent morphism between the adjoint tangent categories.

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$(F,\alpha):(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}')$

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Lemma

Consider a lax tangent morphism between two adjunctable tangent categories. Then the opposite of F is also a lax tangent morphism between the adjoint tangent categories.

$$(F,\alpha):(\mathbb{X},\mathbb{T})\longrightarrow(\mathbb{X}',\mathbb{T}'),\quad \mathbb{T}^{\circ}\dashv\mathbb{T},\quad \mathbb{T}'^{\circ}\dashv\mathbb{T}'$$

Lemma

Consider a lax tangent morphism between two $\boxed{\text{adjunctable}}$ tangent categories. Then the opposite of F is also a lax tangent morphism between the adjoint tangent categories.

$$\frac{(F,\alpha):(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}'),\quad \mathbb{T}^{\circ}\dashv\mathbb{T},\quad \mathbb{T}'^{\circ}\dashv\mathbb{T}'}{(F^{op},\alpha^{\circ}):(\mathbb{X}^{op},\mathbb{T}^{\circ})\to(\mathbb{X}'^{op},\mathbb{T}'^{\circ})}$$

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$\alpha:F\mathbb{T}\Rightarrow\mathbb{T}'F$

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$$\alpha: F\mathrm{T}\Rightarrow\mathrm{T}'F$$

$$\alpha^\circ:\mathrm{T}'^\circ F\xrightarrow{\mathrm{T}'^\circ F\eta}\mathrm{T}'^\circ F\mathrm{T}\mathrm{T}^\circ\xrightarrow{\mathrm{T}'^\circ\alpha\mathrm{T}^\circ}\mathrm{T}'^\circ\mathrm{T}'F\mathrm{T}^\circ\xrightarrow{\epsilon' F\mathrm{T}^\circ}F\mathrm{T}^\circ$$

Prop'n

The operation which sends an operad to its geometric tangent category extends to two functors:

$$\mathrm{Geom}^*: \mathrm{Operad}^{op} \longrightarrow \mathrm{TngCat}$$
$$\mathcal{P} \longmapsto \mathrm{Geom}(\mathcal{P})$$
$$(\varphi: \mathcal{P} \to \mathcal{Q}) \longmapsto (\varphi^*, \beta^*): \mathrm{Geom}(\mathcal{Q}) \to \mathrm{Geom}(\mathcal{P})$$

Not isomorphism

Functoriality

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The operation which sends an operad to its geometric tangent category extends to two functors:

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$$(\varphi: \mathcal{P} \to \mathcal{Q}) \longmapsto (\varphi_!, \beta_!): \mathrm{Geom}(\mathcal{P}) \to \mathrm{Geom}(\mathcal{Q})$$

This is an iso!!!

Chapter 4

The enveloping operad

"The coslice of the category of algebras is still a category of algebras."

The enveloping operad

The enveloping operad

$$\mathbf{Operad}^{op} \longrightarrow \mathbf{Cat}$$

indexed category

$$\mathcal{P} \longmapsto \mathbf{Alg}_{\mathcal{P}}$$

$$(\varphi : \mathcal{P} \to \mathcal{Q}) \longmapsto \mathbf{Alg}_{\mathcal{Q}} \xrightarrow{\varphi^*} \mathbf{Alg}_{\mathcal{P}}$$

The enveloping operad

The enveloping operad

$$\frac{\begin{array}{rcl} \mathsf{Operad}^{op} & \longrightarrow & \mathsf{Cat} \\ \mathcal{P} & \longmapsto & \mathsf{Alg}_{\mathcal{P}} \\ (\varphi : \mathcal{P} \to \mathcal{Q}) & \longmapsto & \mathsf{Alg}_{\mathcal{Q}} \xrightarrow{\varphi^*} \mathsf{Alg}_{\mathcal{P}} \end{array}}{\mathsf{Alg} \longrightarrow \mathsf{Operad}}$$

indexed category

Grothendieck construction

The enveloping operad

The enveloping operad

$$\mathrm{Operad}^{op} \longrightarrow \mathrm{Cat}$$

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Grothendieck construction

$$\mathrm{Alg} \longrightarrow \mathrm{Operad}$$

$$(\mathcal{P}, A) \longmapsto \mathcal{P}$$

$$(\mathcal{P}, A) \longrightarrow (\mathcal{Q}, B) : \mathcal{P} \xrightarrow{\varphi} \mathcal{Q}, \quad A \xrightarrow{f} \varphi^* B$$

The enveloping operad

The enveloping operad

$$\mathrm{Operad}^{op} \longrightarrow \mathrm{Cat}$$
$$\mathcal{P} \longmapsto \mathrm{Alg}_{\mathcal{P}}$$
$$(\varphi : \mathcal{P} \to \mathcal{Q}) \longmapsto \mathrm{Alg}_{\mathcal{Q}} \xrightarrow{\varphi^*} \mathrm{Alg}_{\mathcal{P}}$$

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$$(\mathcal{P}, A) \longrightarrow (\mathcal{Q}, B) : \mathcal{P} \xrightarrow{\varphi} \mathcal{Q}, \quad A \xrightarrow{f} \varphi^* B$$

$$\mathrm{Init} : \mathrm{Operad} \longrightarrow \mathrm{Alg}$$
$$\mathcal{P} \longmapsto (\mathcal{P}, \mathcal{P}(0))$$

Initial $\mathcal{P}$-algebra

$$(\varphi : \mathcal{P} \to \mathcal{Q}) \longmapsto (\mathcal{P}, \mathcal{P}(0)) \longrightarrow (\mathcal{Q}, \mathcal{Q}(0))$$
$$\varphi : \mathcal{P} \to \mathcal{Q}$$
$$\mathcal{P}(0) \xrightarrow{\varphi_0} \varphi^* \mathcal{Q}(0)$$

The enveloping operad

The enveloping operad

$$\mathbf{Operad}^{op} \longrightarrow \mathbf{Cat}$$
$$\mathcal{P} \longmapsto \mathbf{Alg}_{\mathcal{P}}$$
$$(\varphi : \mathcal{P} \to \mathcal{Q}) \longmapsto \mathbf{Alg}_{\mathcal{Q}} \xrightarrow{\varphi^*} \mathbf{Alg}_{\mathcal{P}}$$

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$$\mathrm{Env} : \mathbf{Alg} \rightleftarrows \mathbf{Operad} : \mathrm{Init}$$
$$(\mathcal{P}, A) \longmapsto \mathcal{P}^{(A)}$$

The enveloping operad

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$$\mathrm{Operad}^{op} \longrightarrow \mathrm{Cat}$$
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$\mathcal{P}^{(A)}(n) \qquad (\mu; a_1, \dots, a_k \mid \qquad$ + Relations

The enveloping operad

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$$\mathrm{Operad}^{op} \longrightarrow \mathrm{Cat}$$
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$$\mathcal{P}^{(A)}(n) \qquad (\mu ; a_1, \dots, a_k |$$

$\mu \in \mathcal{P}(n+k)$

A + Relations

The enveloping operad

Lemma

The category of algebras of the enveloping operad is the coslice category of algebras of the original operad.

The enveloping operad

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The category of algebras
of the enveloping operad is
the coslice category of algebras
of the original operad.

$$\mathcal{P}^{(A)}B \xrightarrow{\theta} B$$

$$(\mu; a_1, \dots, a_k \mid (b_1, \dots, b_n)$$

$$A \xrightarrow{\sim} \mathcal{P}^{(A)}(0)$$

The enveloping operad

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The enveloping operad

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The category of algebras of the enveloping operad is the coslice category of algebras of the original operad.

$\mathcal{P}$-algebra

$f : A \longrightarrow B$

$\mathcal{P}^{(A)} B \xrightarrow{\theta} B$

$(\mu; a_1, \dots, a_k \mid (b_1, \dots, b_n)$

$(\mu; a_1, \dots, a_k \mid (b_1, \dots, b_n) :=$

$\mu(f(a_1), \dots, f(a_k), b_1, \dots, b_n)$

$A \xrightarrow{\sim} \mathcal{P}^{(A)}(0) \hookrightarrow \mathcal{P}^{(A)} B \xrightarrow{\theta} B$

The enveloping operad

Prop'n

The ring $\mathscr{P}^{(A)}(1)$ is the enveloping algebra of A.

Modules over A are left modules over $\mathscr{P}^{(A)}(1)$.

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+ Relations

$\mathcal{L}ie^{(\mathfrak{g})}(1)$ is the universal enveloping algebra of $\mathfrak{g}$

The enveloping operad

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The ring $\mathscr{P}^{(A)}(1)$ is the enveloping algebra of A.

Modules over A are left modules over $\mathscr{P}^{(A)}(1)$.

$(\mu; a_1, \dots, a_{n-1})$ + Relations

$\mathrm{Lie}^{(\mathfrak{g})}(1)$ is the universal enveloping algebra of $\mathfrak{g}$

$$\frac{M, \quad \mathscr{P}(n) \otimes A^{\otimes n-1} \otimes M \longrightarrow M}{\mathscr{P}^{(A)}(1) \otimes M \longrightarrow M}$$

First contact

Chapter 5

"The geometric tangent category
of the enveloping operad
is the slice of the
geometric tangent category
of the original operad."

Theorem

The geometric tangent category of the enveloping operad is the slice tangent category of the geometric tangent category of the original operad.

Theorem

The geometric tangent category of
the enveloping operad is the slice tangent category
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of the original operad.

$$
\begin{tikzcd}
\mathrm{Operad}^{op} \arrow[r, "\mathrm{Init}^{op}"'] \arrow[d, "\mathrm{Geom}"'] & \mathrm{Alg}^{op} \arrow[d, "\mathrm{Geom}"] \\
\mathrm{tTngCat} \arrow[r, "\mathrm{Term}"] & \mathrm{TngPair}
\end{tikzcd}
$$

Theorem

The geometric tangent category of the enveloping operad is the slice tangent category of the geometric tangent category of the original operad.

$$\begin{tikzcd}
\mathsf{Operad}^{op} \arrow[d, "\mathsf{Geom}"'] \arrow[r, shift right, "\mathsf{Init}^{op}"'] & \mathsf{Alg}^{op} \arrow[l, shift right, "\mathsf{Env}^{op}"'] \arrow[d, "\mathsf{Geom}"] \\
\mathsf{tTngCat} \arrow[r, shift left, "\mathsf{Term}"] & \mathsf{TngPair} \arrow[l, shift left, "\mathsf{Slice}"]
\end{tikzcd}$$

First contact

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Theorem

The geometric tangent category of the enveloping operad is the slice tangent category of the geometric tangent category of the original operad.

$$\begin{tikzcd}
\mathrm{Operad}^{op} \arrow[r, shift right, "\mathrm{Init}^{op}"'] \arrow[d, "\mathrm{Geom}"'] & \mathrm{Alg}^{op} \arrow[l, shift right, "\mathrm{Env}^{op}"'] \arrow[d, "\mathrm{Geom}"] \\
\mathrm{tTngCat} \arrow[r, shift left, "\mathrm{Term}"] & \mathrm{TngPair} \arrow[l, shift left, "\mathrm{Slice}"]
\end{tikzcd}$$

$$\mathrm{Geom}(\mathcal{P}^{(A)}) \longrightarrow \mathrm{Slice}(\mathrm{Geom}(\mathcal{P}), A)$$

First contact

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Theorem

The geometric tangent category of the enveloping operad is the slice tangent category of the geometric tangent category of the original operad.

$$\begin{tikzcd}
\mathrm{Operad}^{op} \arrow[r, shift right, "\mathrm{Init}^{op}"'] \arrow[d, "\mathrm{Geom}"'] & \mathrm{Alg}^{op} \arrow[l, shift right, "\mathrm{Env}^{op}"'] \arrow[d, "\mathrm{Geom}"] \\
\mathrm{tIngCat} \arrow[r, shift left, "\mathrm{Term}"] & \mathrm{IngPair} \arrow[l, shift left, "\mathrm{Slice}"]
\end{tikzcd}$$

$$\mathrm{Geom}(\mathscr{P}^{(A)}) \xrightarrow{\sim} \mathrm{Slice}(\mathrm{Geom}(\mathscr{P}), A)$$

Corollary

The differential bundles over an operadic affine scheme are modules over the operadic algebra.

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The differential bundles over an operadic affine scheme are modules over the operadic algebra.

$$\mathrm{DBun}(\mathrm{Geom}(\mathfrak{P}); A) \cong \mathrm{DObj}^{\bullet}(\mathrm{Geom}(\mathfrak{P})/A)$$

Differential bundles are Differential objects in the slice tangent category

Corollary

The differential bundles over an operadic affine scheme are modules over the operadic algebra.

$$\mathrm{DBund}(\mathrm{Geom}(\mathscr{P});A) \cong \mathrm{DObj}^{\bullet}(\mathrm{Geom}(\mathscr{P})/A) \cong$$

$$\cong \mathrm{DObj}^{\bullet}(\mathrm{Geom}(\mathscr{P}^{(A)}))$$

$$\mathrm{Slice}(\mathrm{Geom}(\mathscr{P});A) \cong \mathrm{Geom}(\mathrm{Env}(\mathscr{P};A))$$

Corollary

The differential bundles over an operadic affine scheme are modules over the operadic algebra.

$$\mathrm{DBund}(\mathrm{Geom}(\mathcal{P}); A) \cong \mathrm{DObj}^{\bullet}(\mathrm{Geom}(\mathcal{P})/A) \cong$$

$$\cong \mathrm{DObj}(\mathrm{Geom}(\mathcal{P}^{(A)})) \cong \mathrm{Mod}^{op}(\mathcal{P}^{(A)}(1))$$

Diff. objects in $\mathrm{Geom}(Q)$ are modules over $Q(1)$

Corollary

The differential bundles over an operadic affine scheme are modules over the operadic algebra.

$$\mathbb{D}\mathrm{Bund}(\mathrm{Geom}(\mathcal{P});A) \cong \mathbb{D}\mathrm{Obj}^{\bullet}(\mathrm{Geom}(\mathcal{P})/A) \cong$$

$$\cong \mathbb{D}\mathrm{Obj}(\mathrm{Geom}(\mathcal{P}^{(A)})) \cong \mathrm{Mod}^{op}(\mathcal{P}^{(A)}(1)) \cong$$

$$\cong \mathbb{M}\mathrm{od}^{op}(\mathcal{P};A)$$

$\mathcal{P}^{(A)}(1)$ enveloping algebra of A

Diff. objects in $\mathrm{Geom}(\mathcal{Q})$ are modules over $\mathcal{Q}(1)$

The end.

Thanks.

The differential bundles
of the geometric tangent category of an operad.
https://arxiv.org/abs/2310.18174