



Dalhousie University
Mathematics & Statistics

The tangent categories of algebras over an Operad

Joint work with Sacha Ikonicoff & Jean-Simon Pacaud Lemay

Marcello Lanfranchi

Supervisors: Dorette Pronk * Geoffrey Cruttwell

Many geometries, one language

There are many different kinds of geometries

Many geometries, one language

Algebraic geometry

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Algebraic geometry

There are many different kinds of geometries

Differential geometry

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Algebraic geometry

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Differential geometry

Synthetic differential geometry

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All of them share some notion of differentiability...

Many geometries, one language

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Is there a common language to describe them all?

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All of them share some notion of differentiability...

Is there a common language to describe them all?

Could this language be tangent category theory?

Many geometries, one language

This is a tough question... and we don't have an answer

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BUT

Many geometries, one language

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*The main
result of
this talk*

BUT

Operads generate tangent categories

Many geometries, one language

This is a tough question... and we don't have an answer

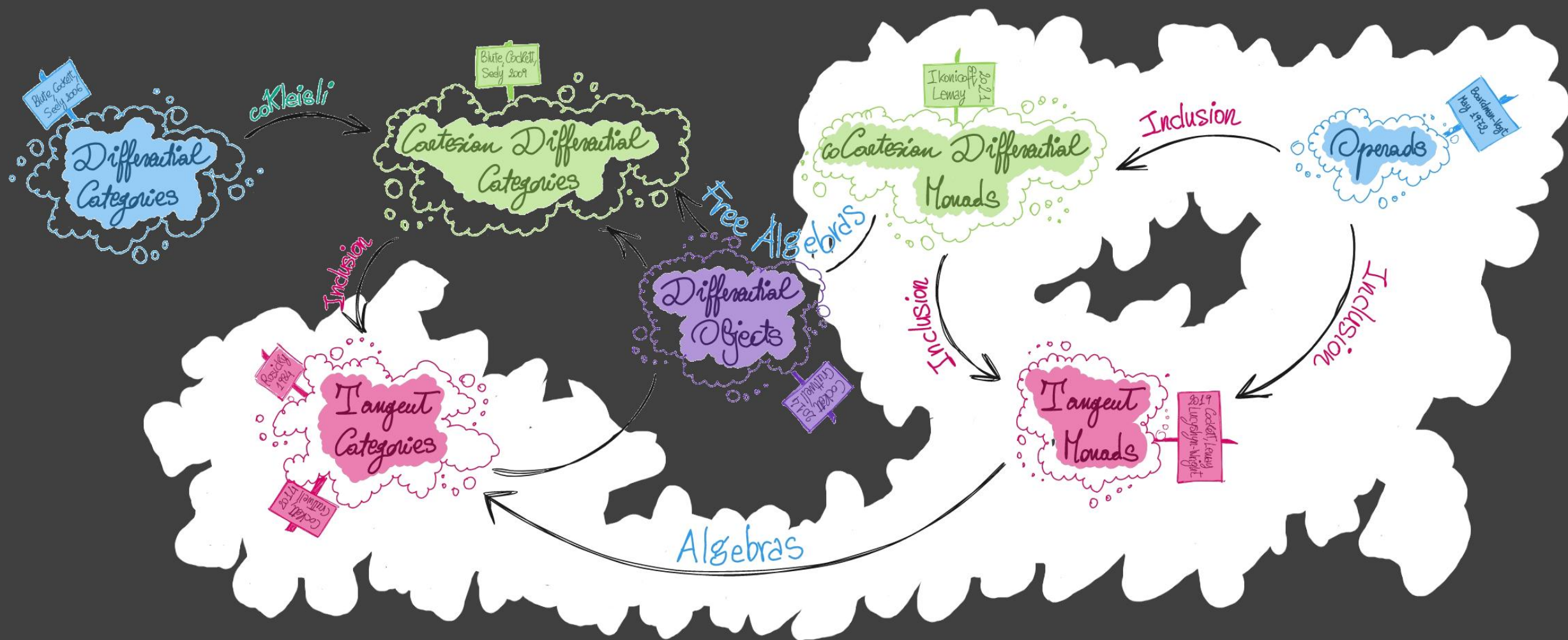
*The main
result of
this talk*

BUT

Operads generate geometrical theories

new models of geometries described with tangent categories

The world of differential categories

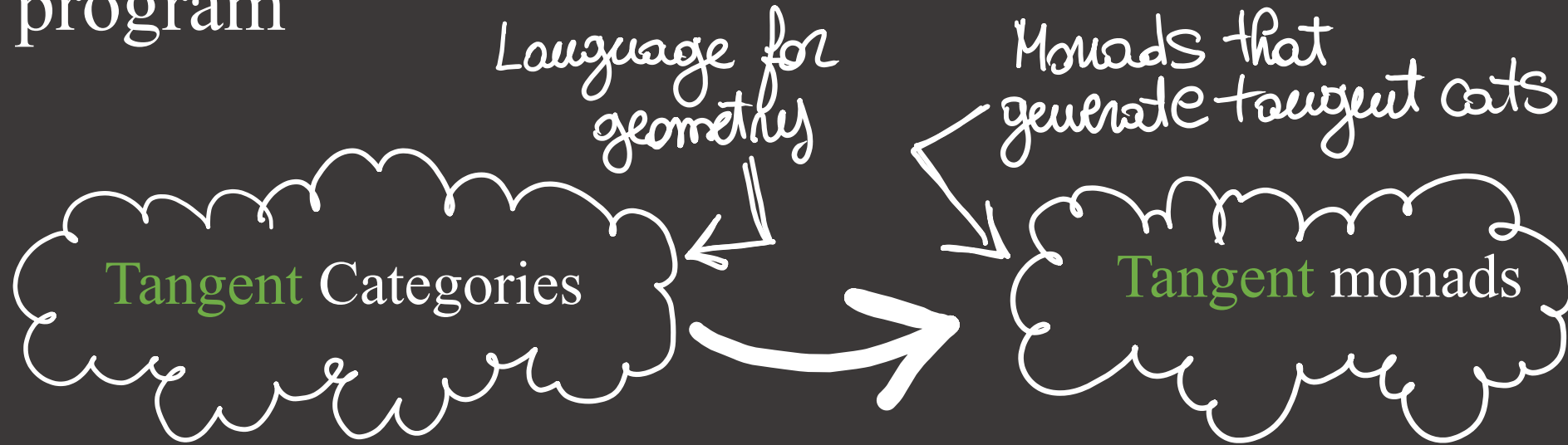


The program

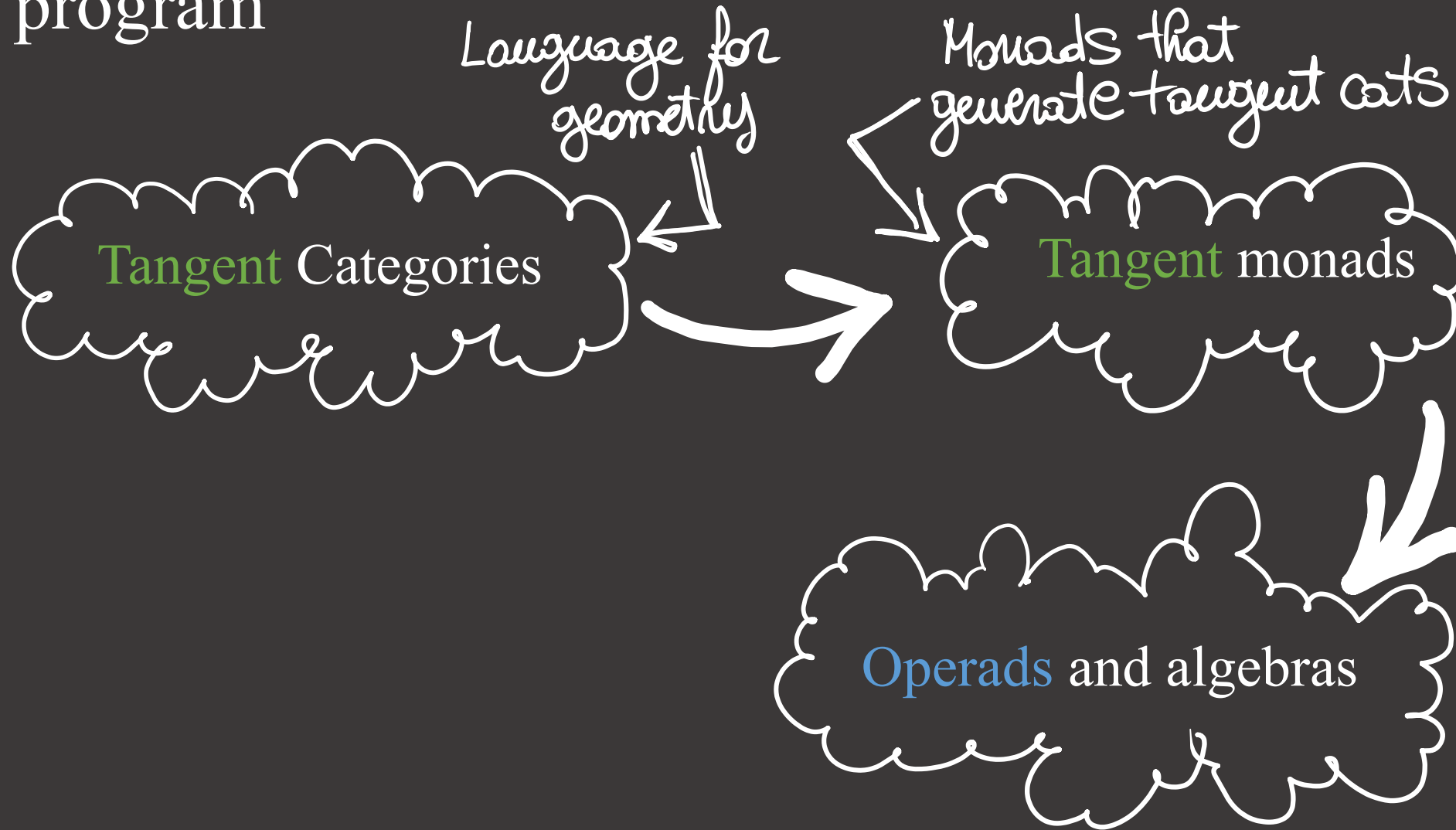
*Language for
geometry*



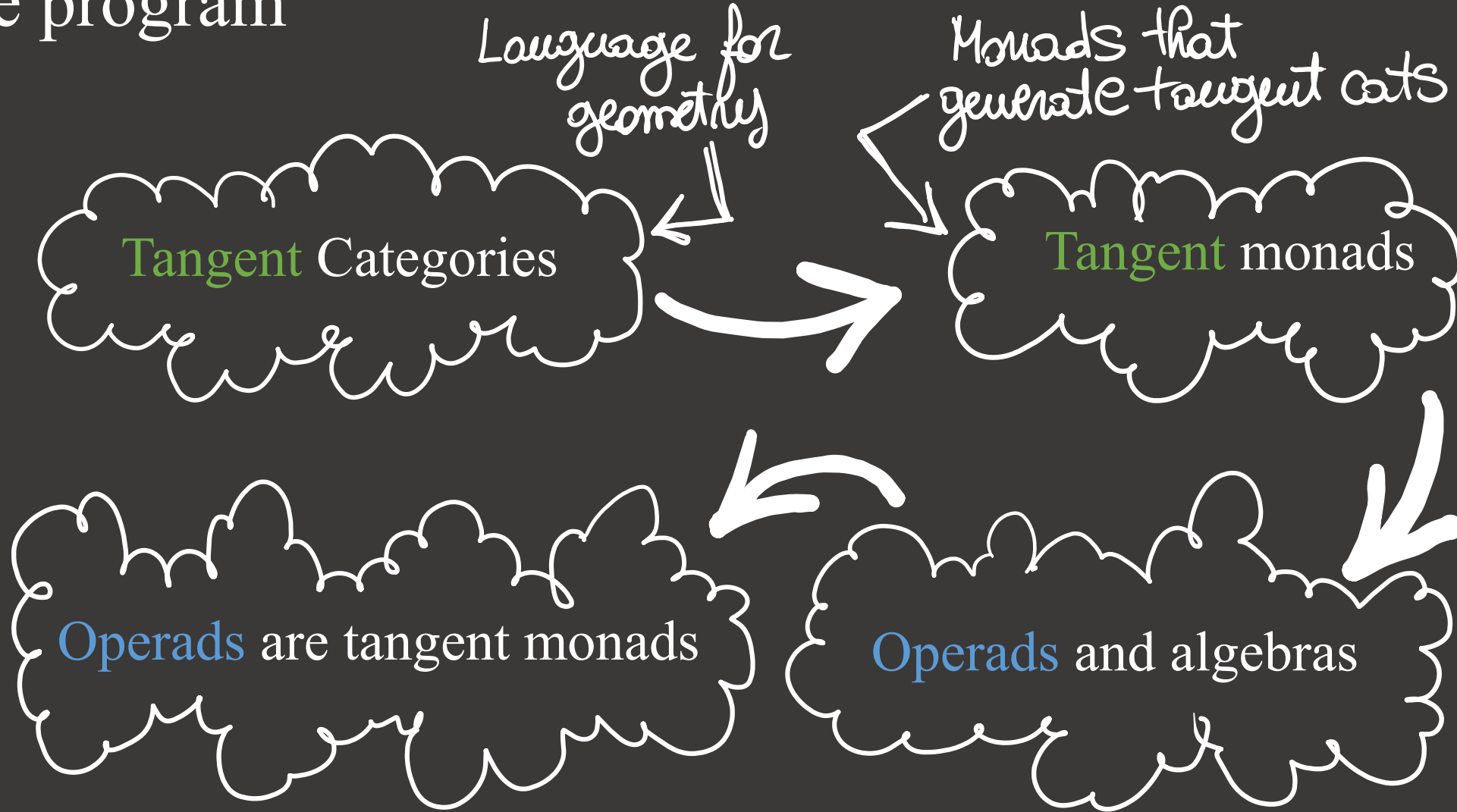
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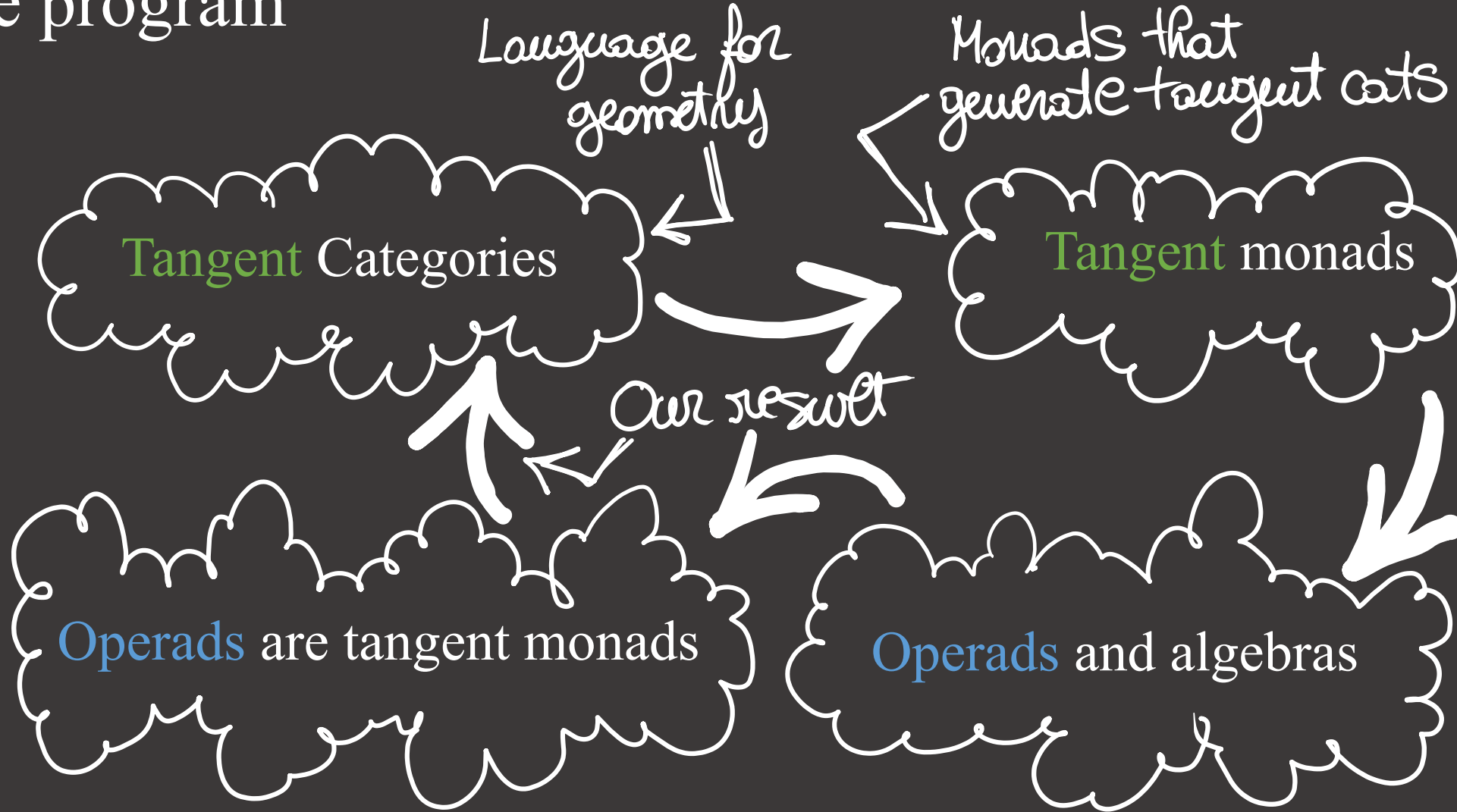
The program



The program



The program



Tangent categories

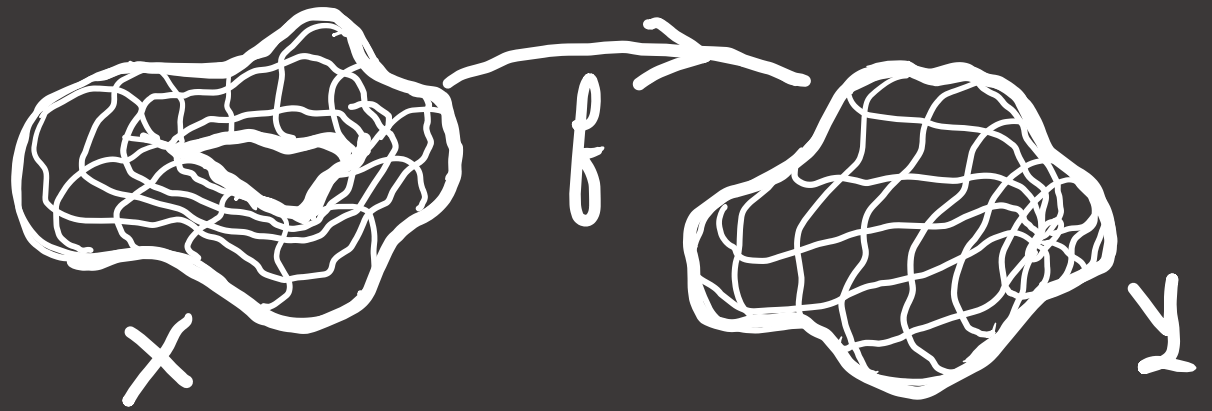
Rosicky 1984
Cockett, Cruttwell 2014

“Tangent category theory
is a categorical language
for differential geometry.”



Tangent categories

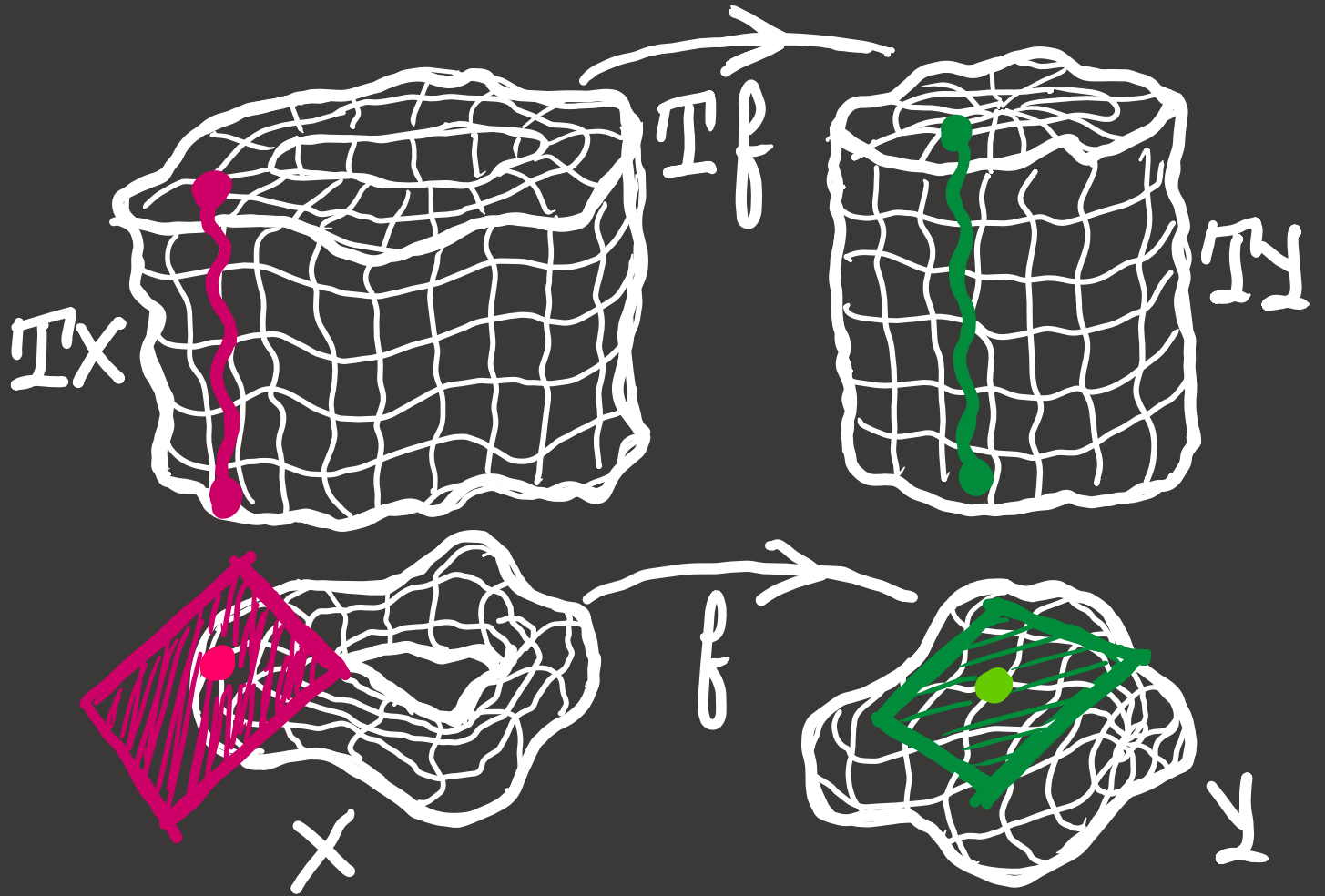
A category



Tangent categories

A category

An endofunctor T

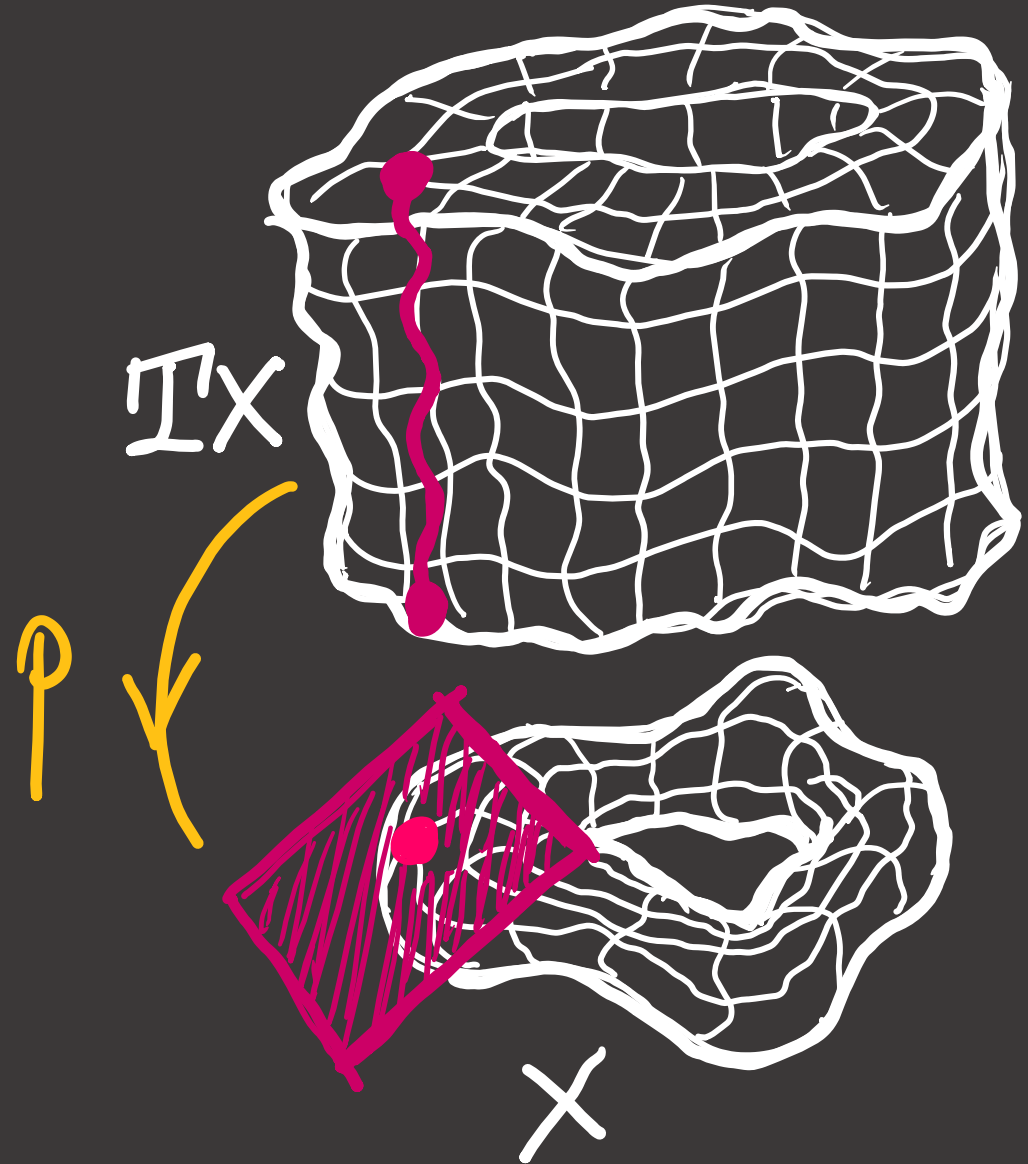


Tangent categories

A category

An endofunctor T

A projection p



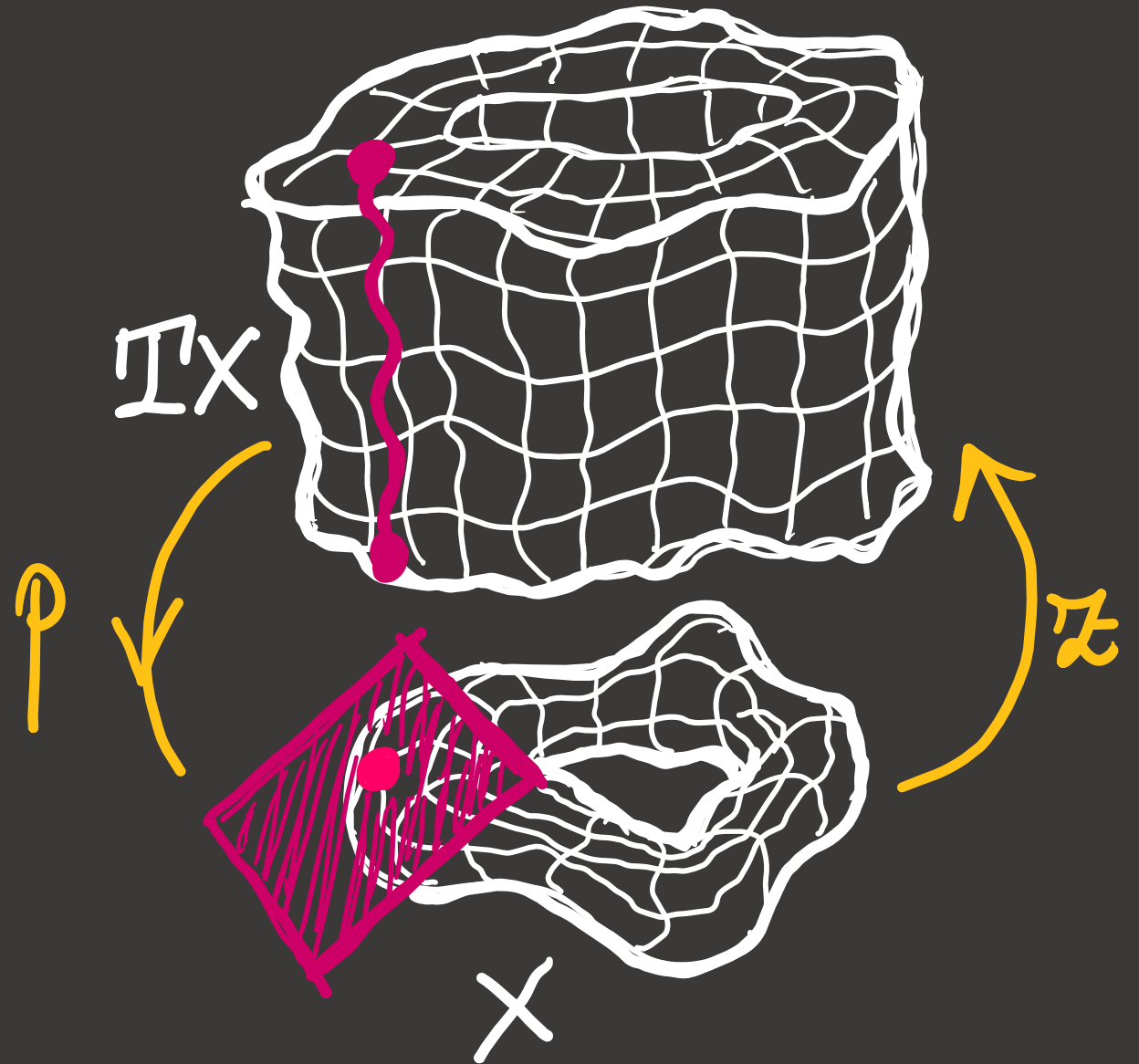
Tangent categories

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A zero morphism z



Tangent categories

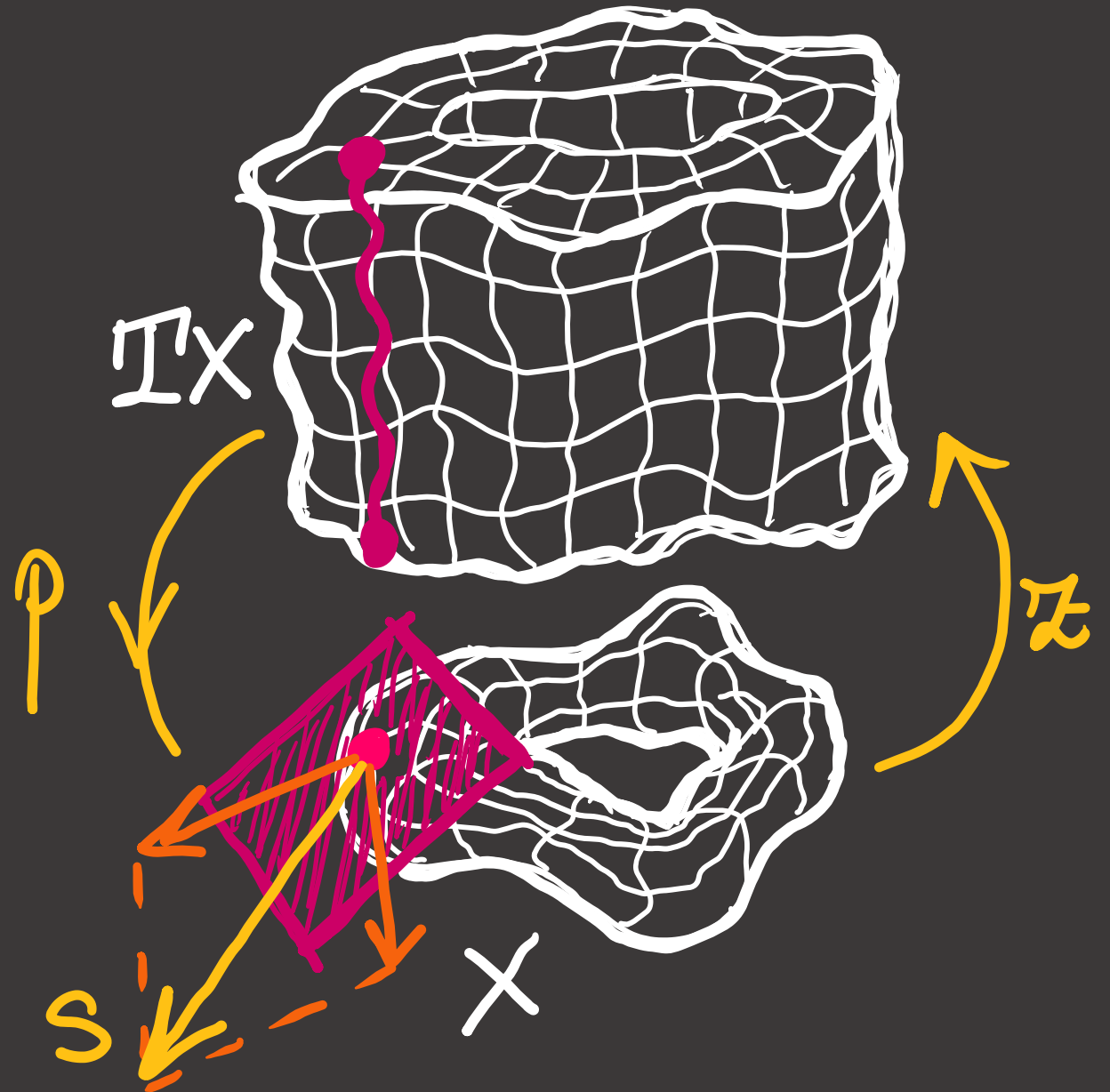
A category

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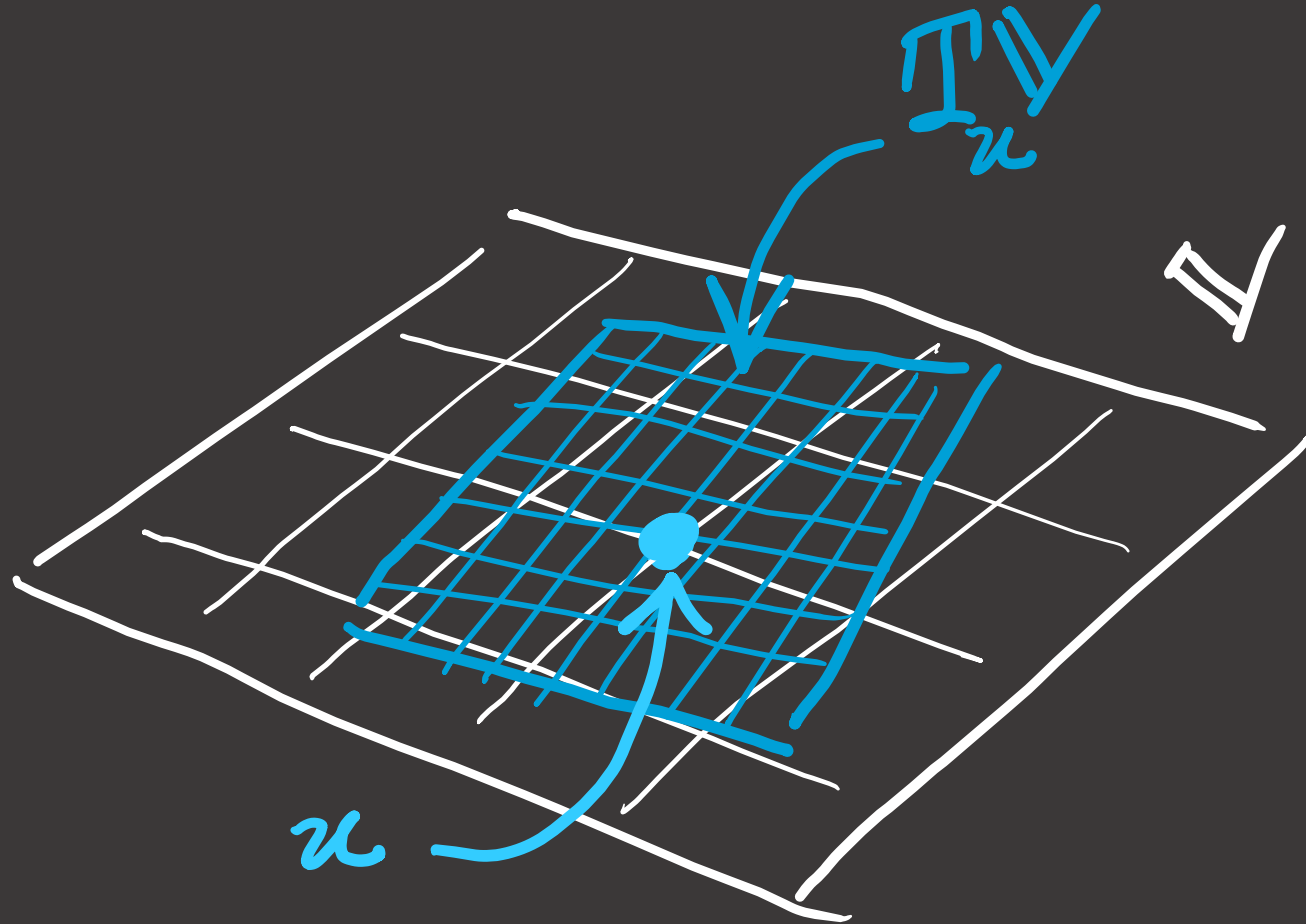
A zero morphism z

A sum morphism s



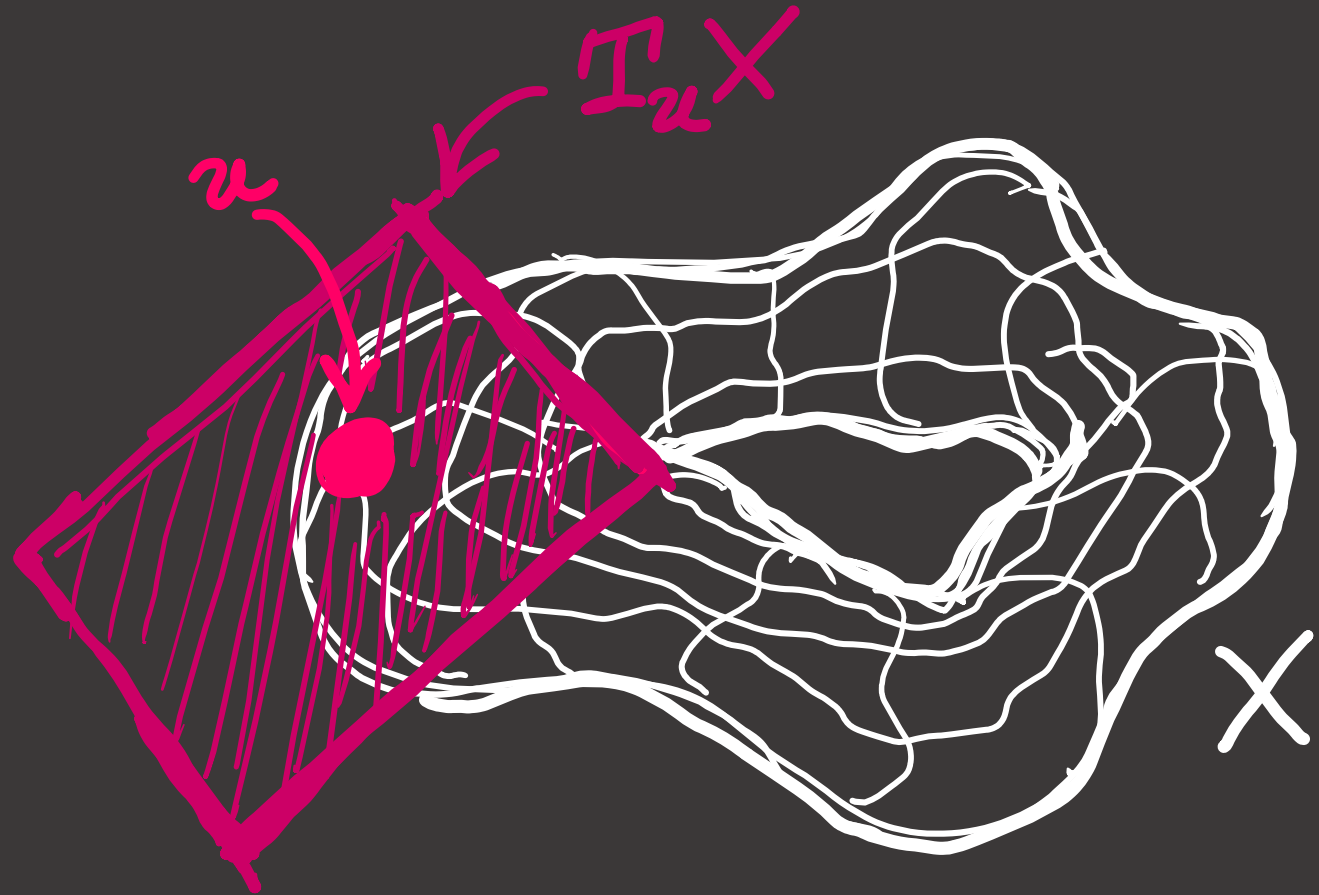
Tangent categories

A vertical lift 1



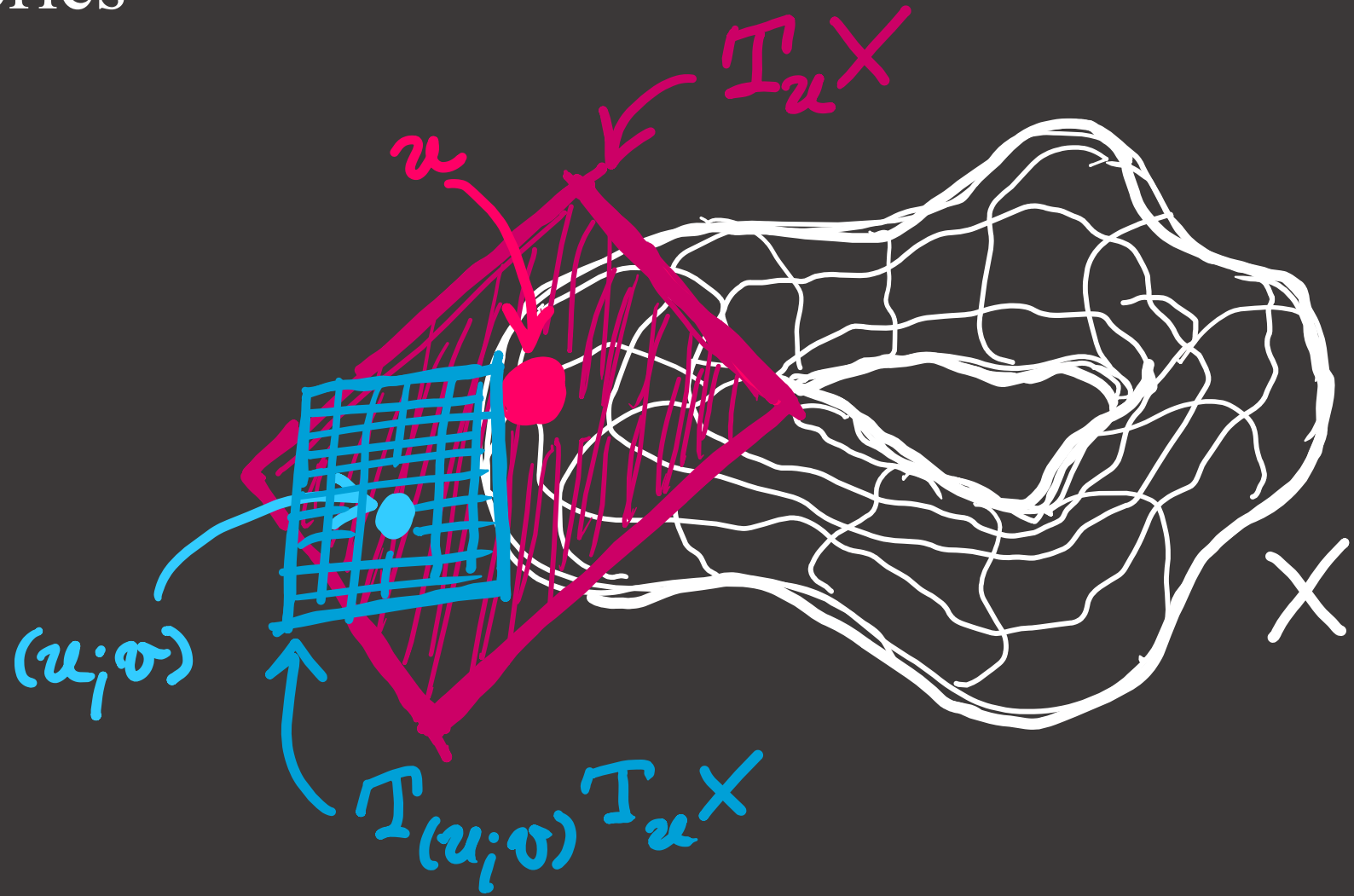
Tangent categories

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Tangent categories

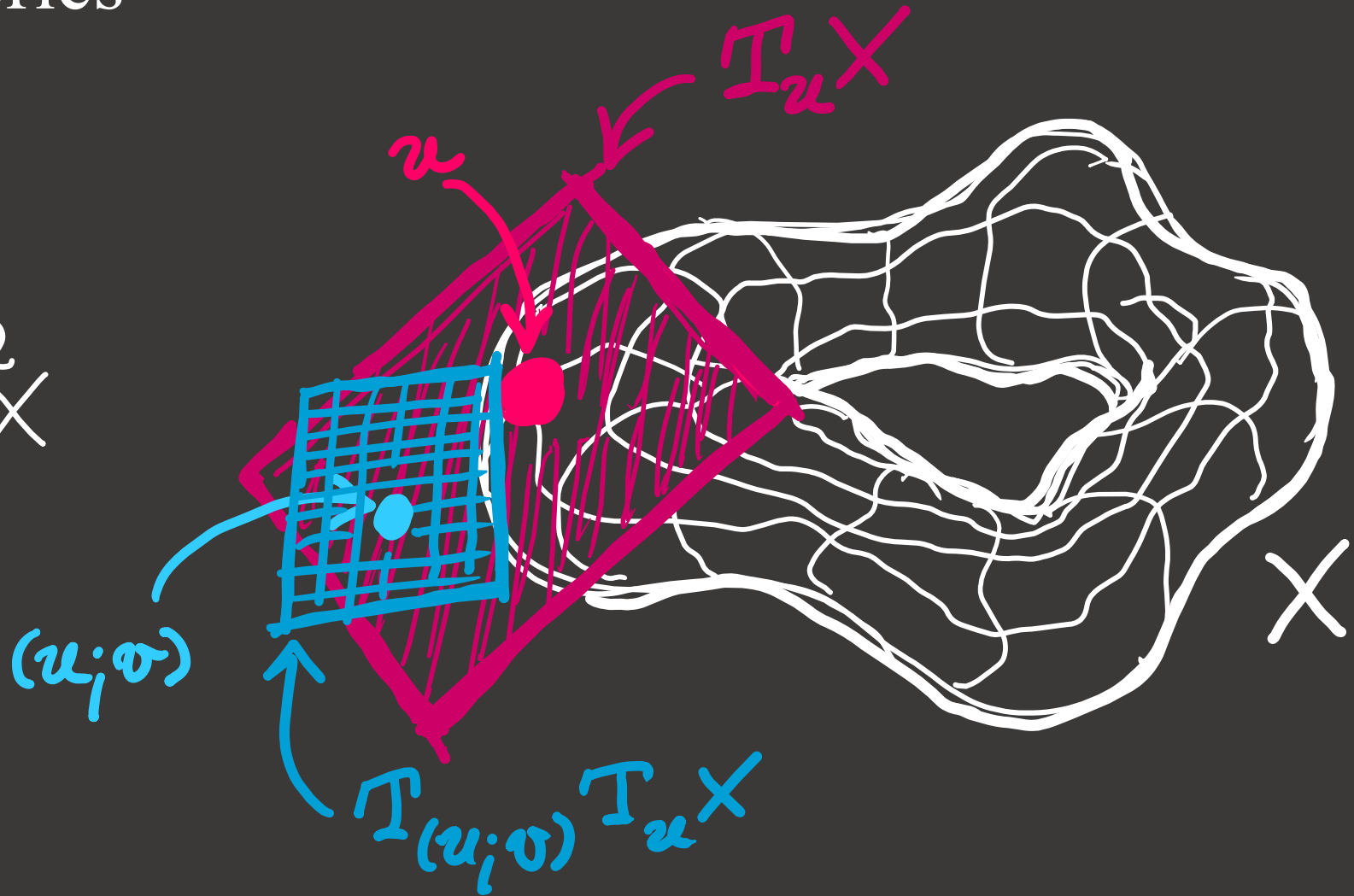
A vertical lift 1



Tangent categories

A vertical lift 1

$$e: TX \rightarrow T^2X$$



Tangent categories

A canonical flip \circlearrowright


💡 Idea!

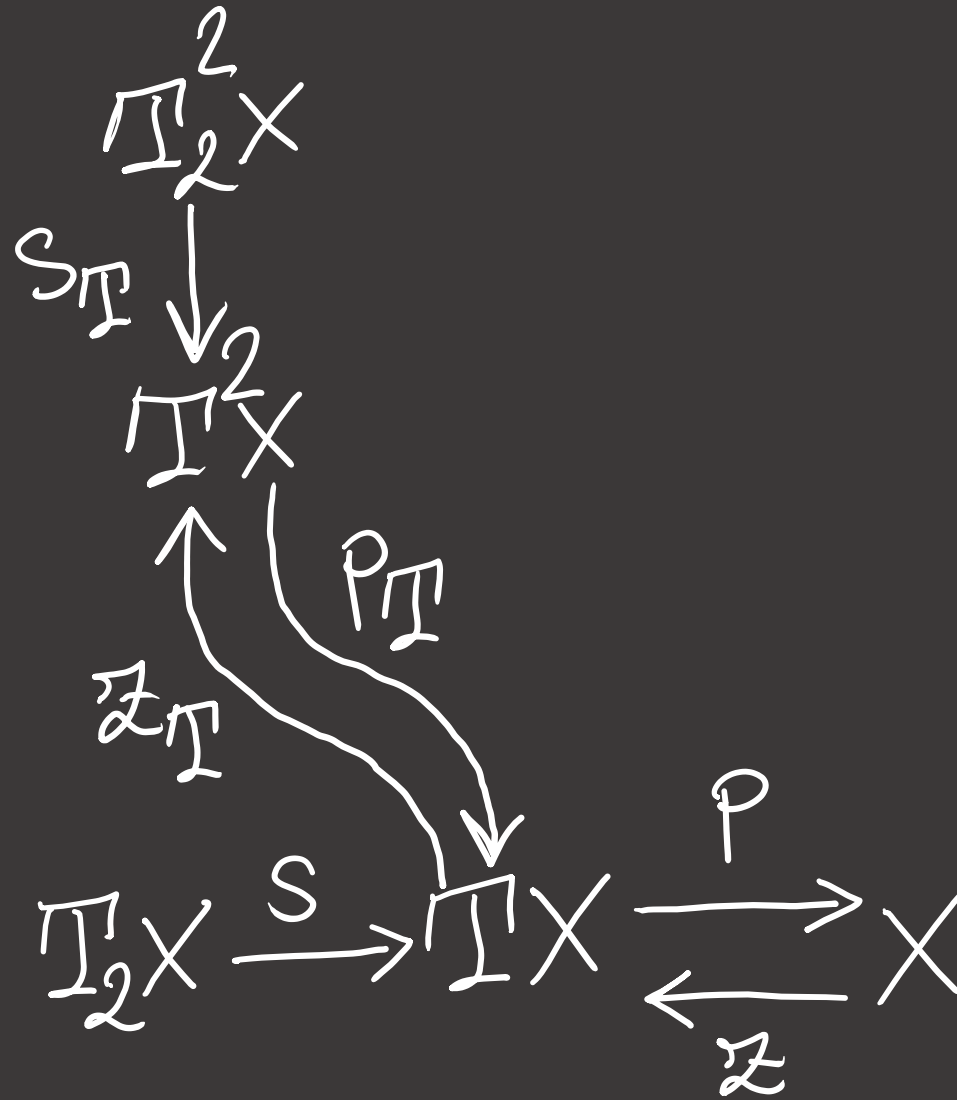
Symmetry of
the Hessian matrix:

$$\partial_{\mu} \partial_{\sigma} = \partial_{\sigma} \partial_{\mu}$$

Tangent categories

A canonical flip \circlearrowleft

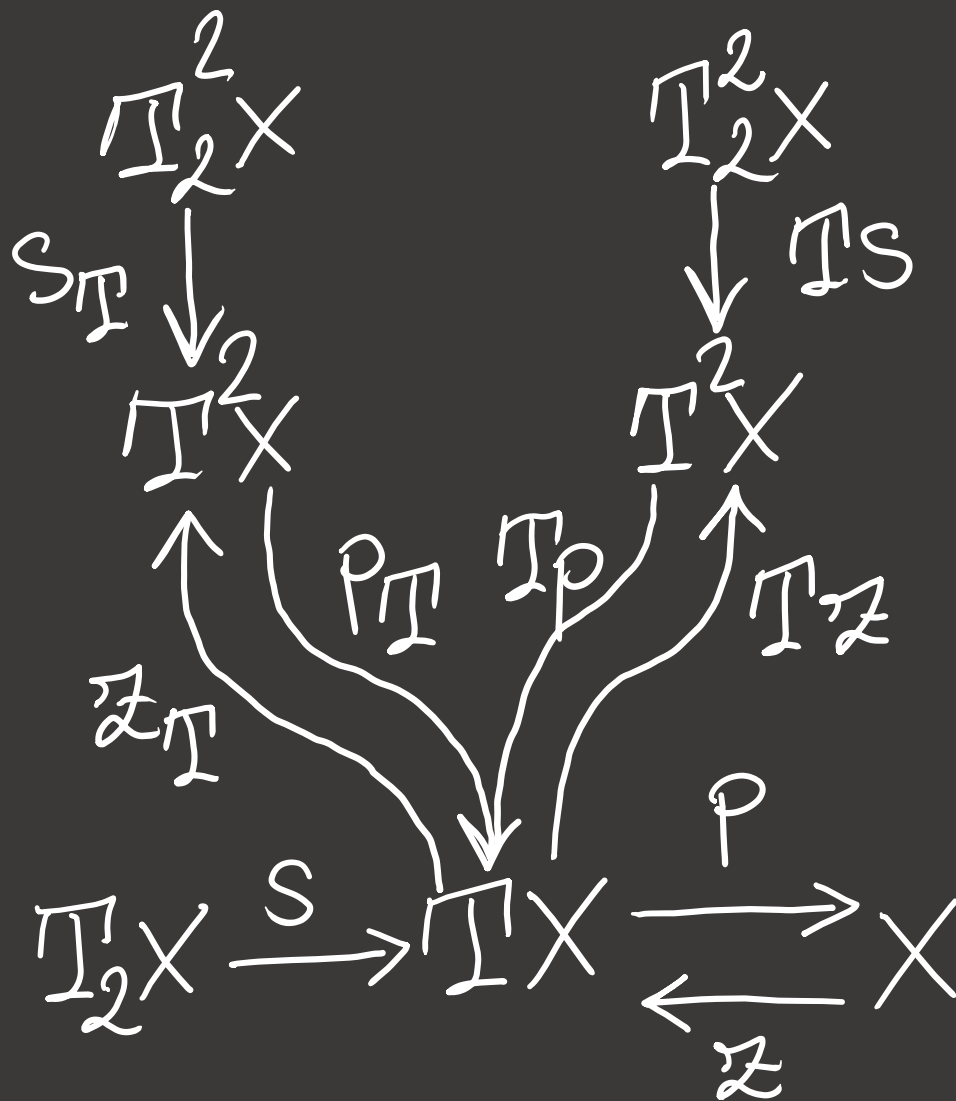
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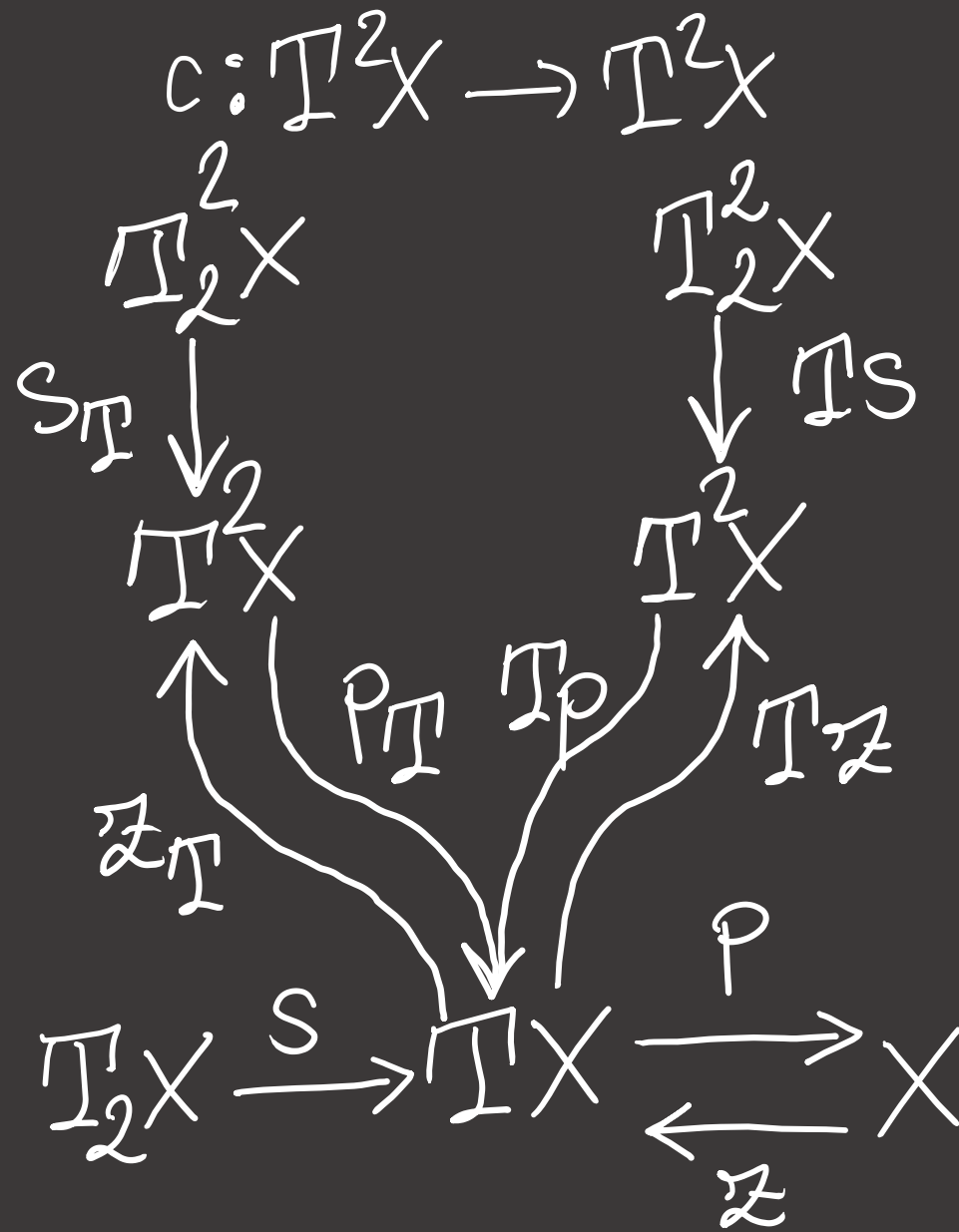
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Tangent categories

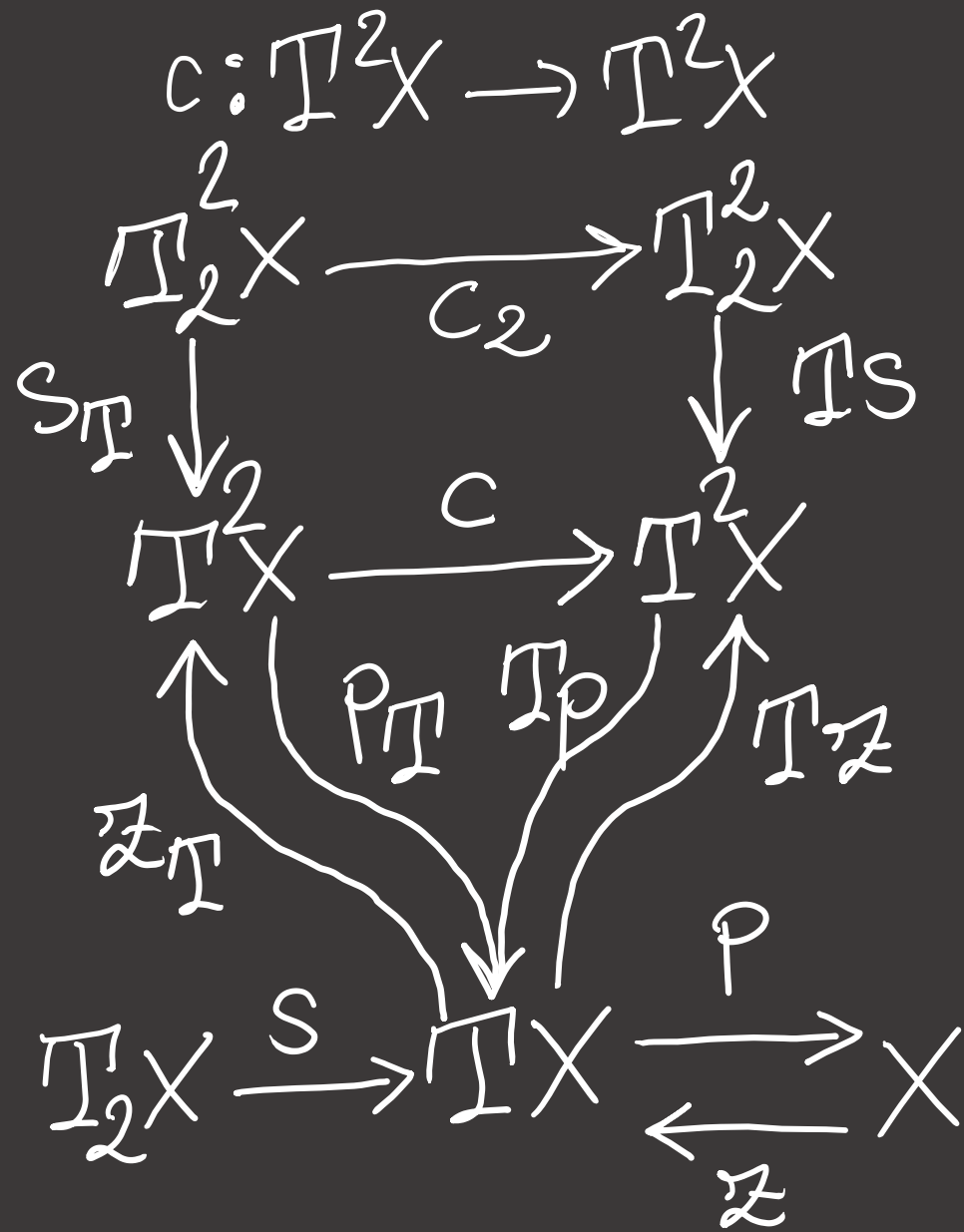
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Tangent categories

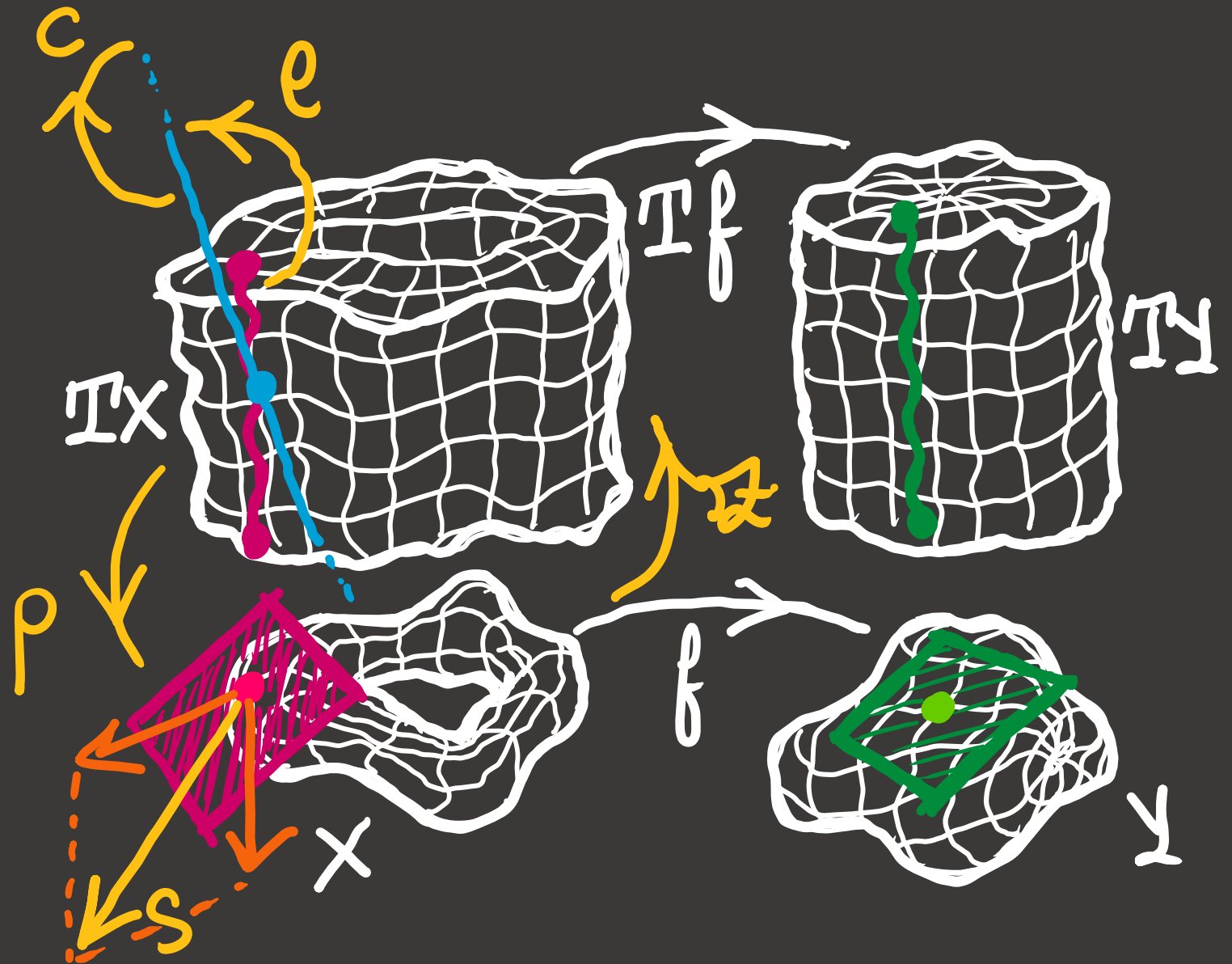
A canonical flip \circlearrowright



Idea!
 Symmetry of
 the Hessian matrix:
 $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$

Tangent categories

Differential geometry



Tangent categories

Categories with biproducts

Products \cong ∞ Products

Every category with biproducts has a canonical tangent structure

$$\mathbb{T}X := X \oplus X$$

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

“Tangent monads are machines that produce tangent categories.”



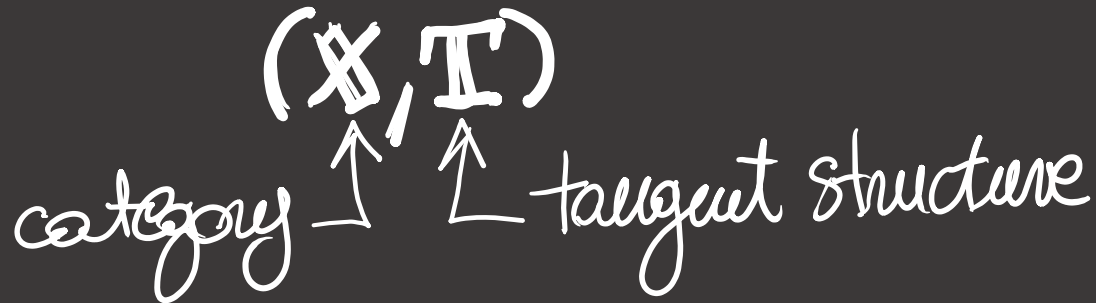
Tangent monads

A tangent monad is a monad in the 2-category of tangent categories

Tangent monads

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A tangent category

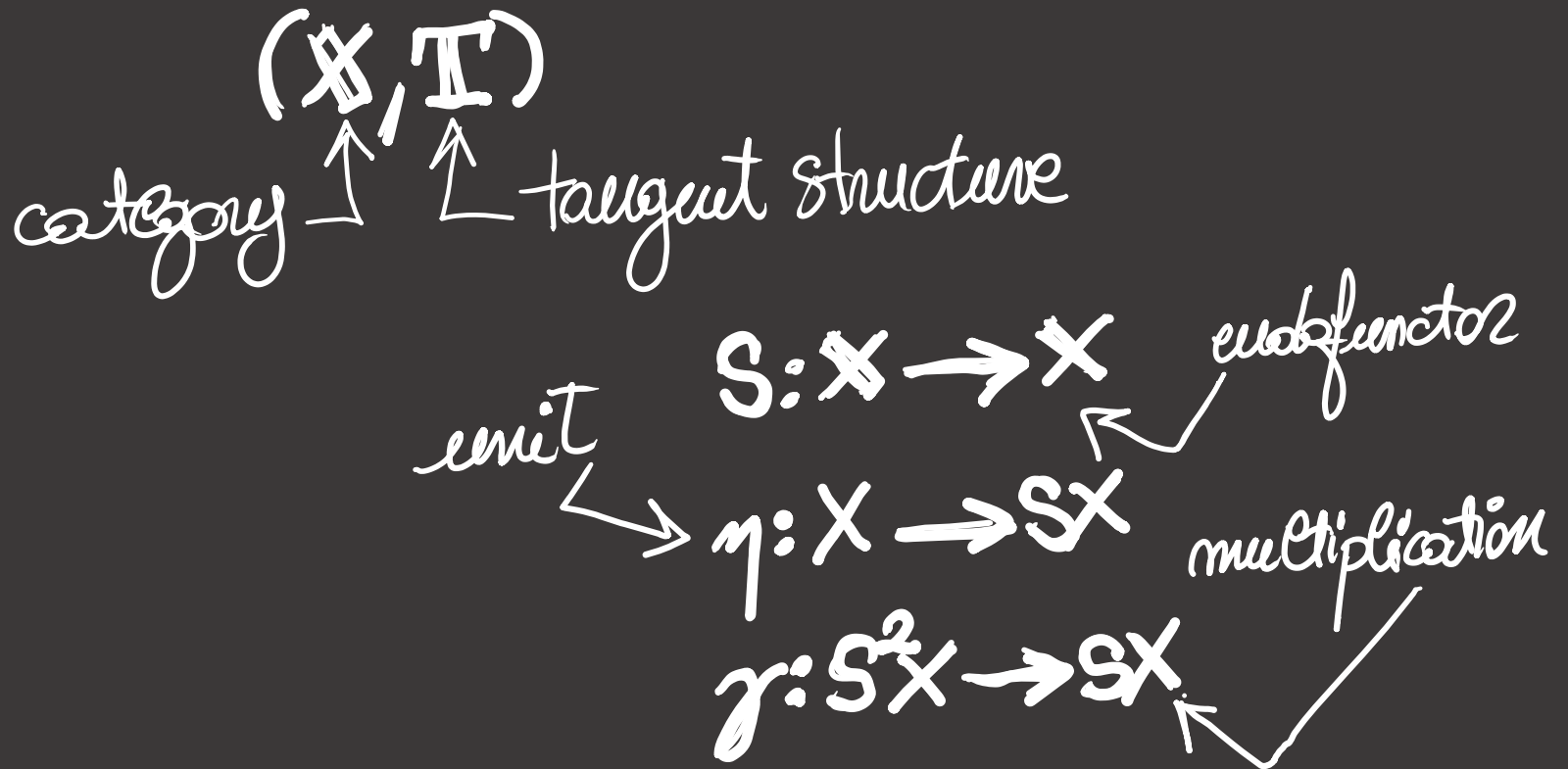


Tangent monads

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A tangent category

A monad



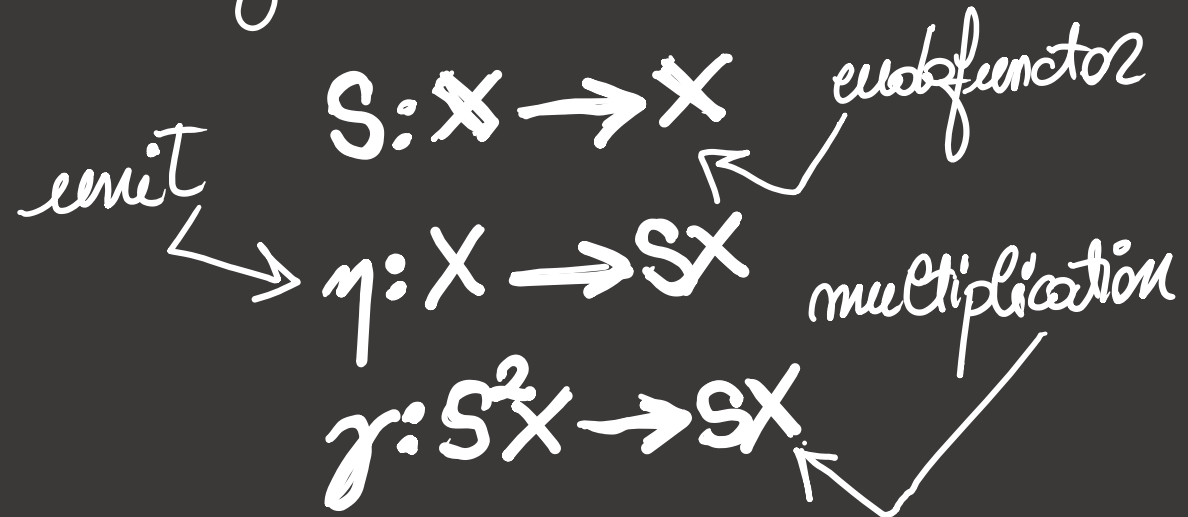
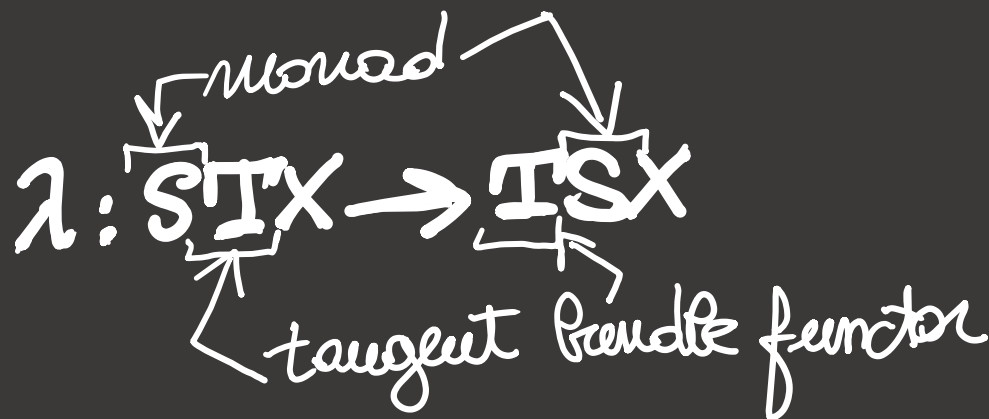
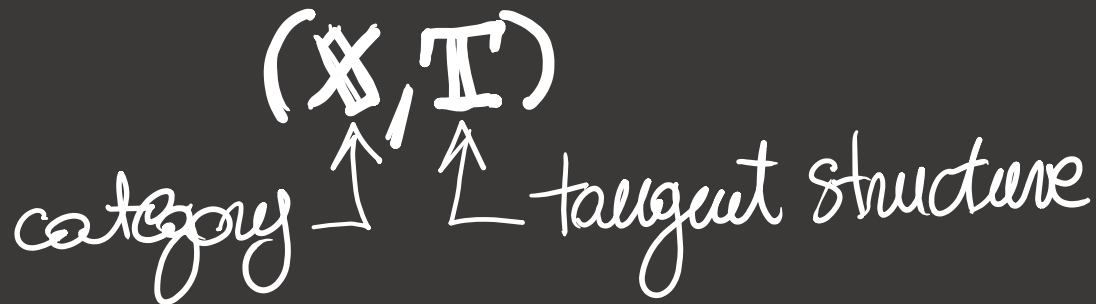
Tangent monads

A tangent monad is a monad in the 2-category of tangent categories

A tangent category

A monad

A distributive law



Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020



Theorem.

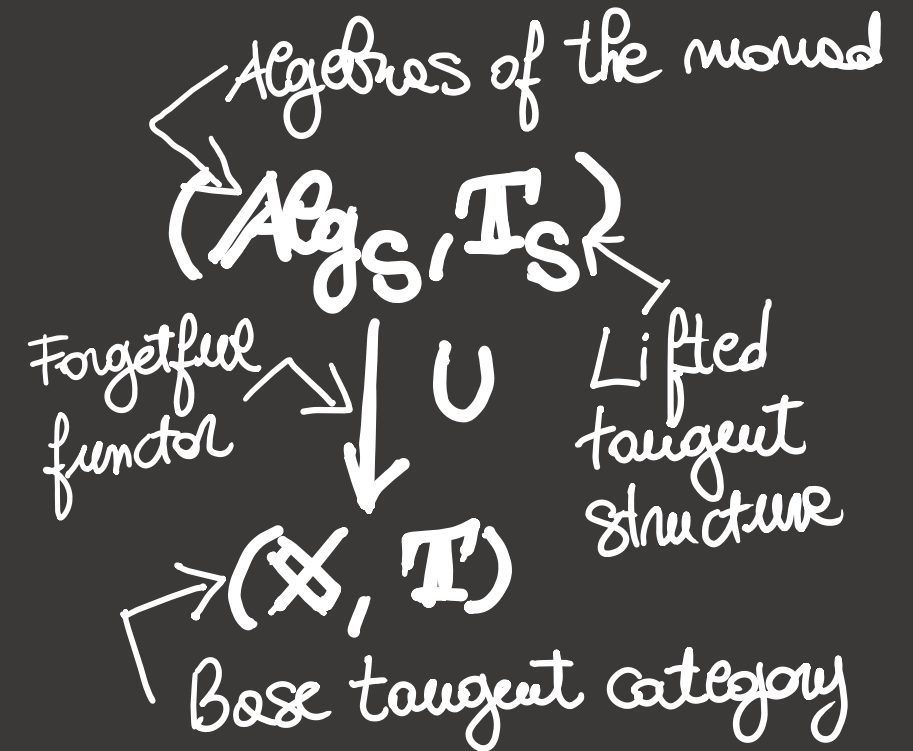
The Eilenberg-Moore category of a tangent monad is a tangent category.

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.

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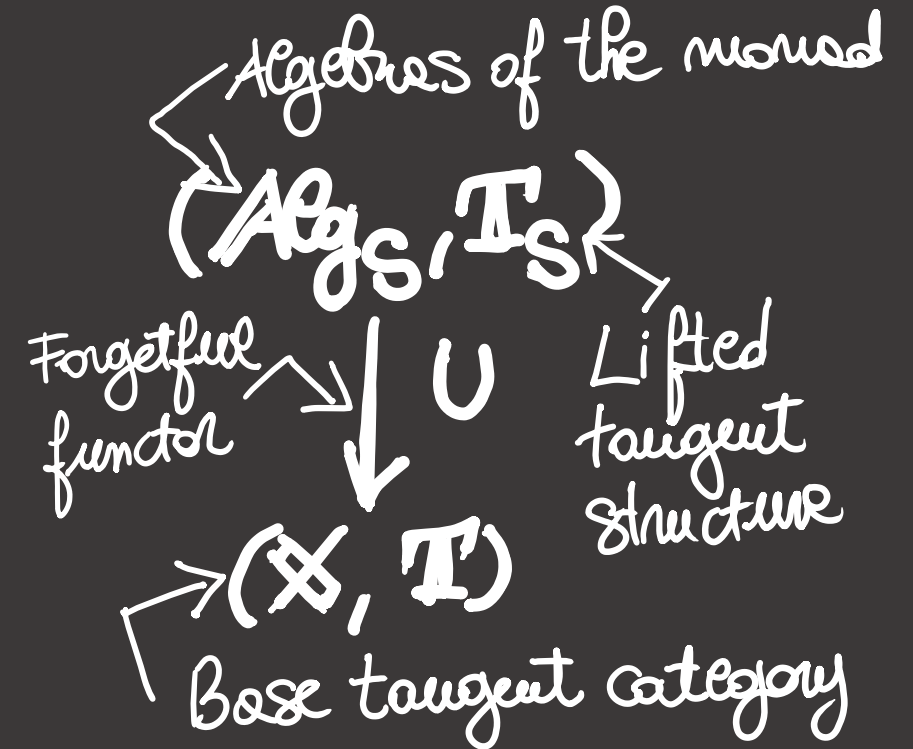


Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.

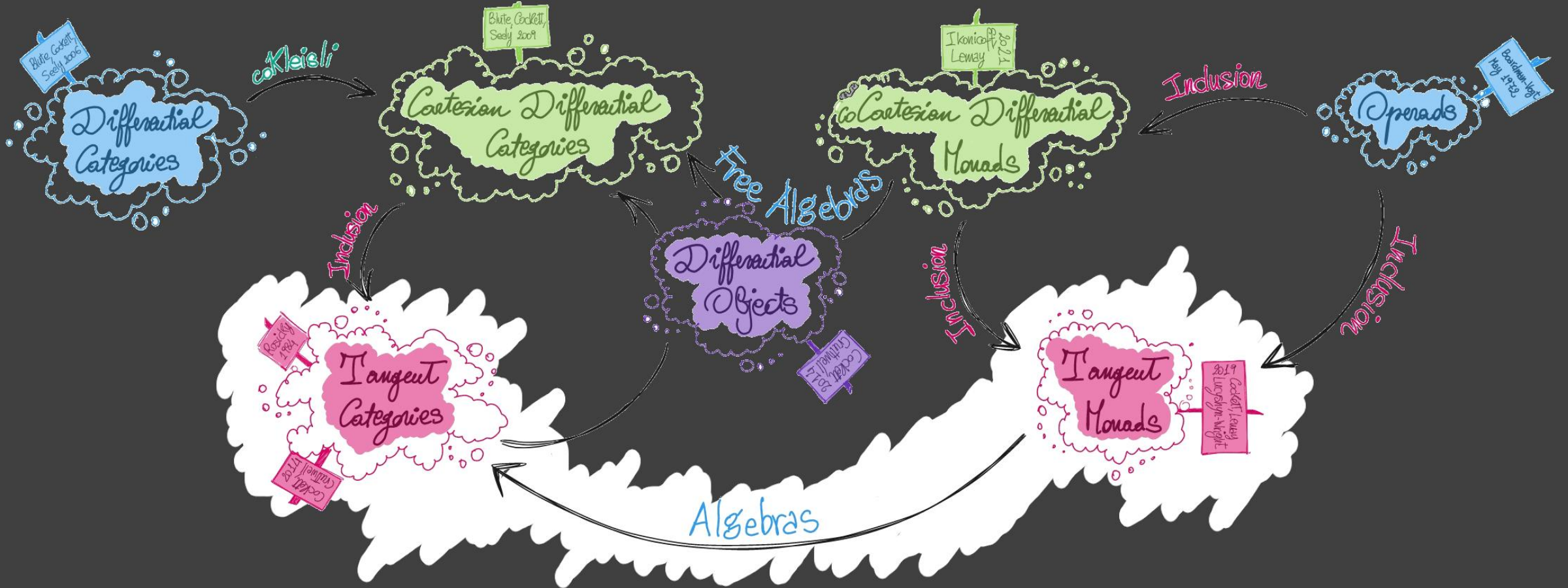
The Eilenberg-Moore category of a tangent monad is a tangent category.



$$\mathbb{T}_S(SA \xrightarrow{\alpha} A) := STA \xrightarrow{\lambda} TSA \xrightarrow{\mathbb{T}\alpha} \mathbb{T}A$$

\swarrow S-algebra
 \nwarrow Distributive law

Tangent monads



Tangent monads

Ikonciff, Lemay 2021

Ikonciff, Lanfranchi, Lemay 2023

Products
||
coProducts

Theorem.

Let (X, B) be the tangent category of biproducts. A tangent monad (S, λ) over (X, B) is equivalent to a **coCartesian differential monad**, that is a monad S equipped with a differential combinator:

$$\partial: SX \rightarrow S(X \oplus X)$$

Tangent monads



Theorem.

Let X be the category with biproducts. If (S, ∂) is a cCDM over X , then Alg_S is a tangent category.

Theorem.

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- Moreover, if Alg_S has reflexive coequalizers then also Alg_S^{op} is a tangent category.

Operads and algebras

“Operads are machines that produce algebraic objects.”



Operads and algebras

We just need Biproducts
and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)



Operads and algebras

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Take the symmetric monoidal category of vector spaces (this can be generalized)

An operad is:

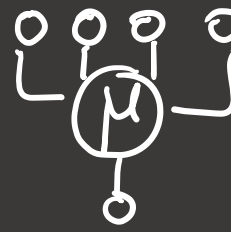


Operads and algebras

We just need Biproducts
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Take the symmetric monoidal category of vector spaces (this can be generalized)

An operad is:

- $\{\mathcal{P}(n)\}$
 - $\begin{array}{c} \circ \\ | \\ \text{Vector spaces} \\ | \\ \text{A unit} \\ \circ \end{array}$
- 

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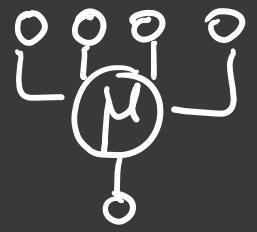


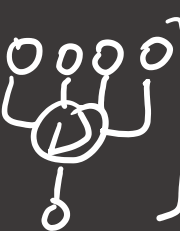
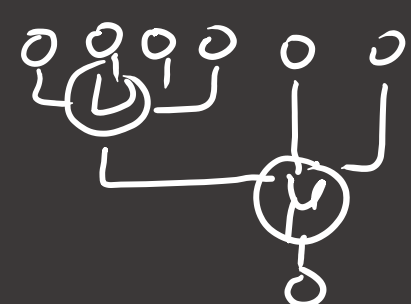
- $\{\mathcal{P}(n)\}$
 - $\mathcal{P}(1) = \{1\}$
 - $\gamma_{\mathcal{R}} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] = \text{Diagram 3}$
- Diagram 1: A circle with '1' inside, one input at the bottom, one output at the top.
- Diagram 2: A circle with 'μ' inside, four inputs at the top, one output at the bottom.
- Diagram 3: A circle with 'μ' inside, one input at the bottom, four outputs at the top. The top two outputs are connected to a circle with 'μ' inside, which has two inputs at the top and one output at the bottom.
- Annotations:
- Vector spaces (with arrow pointing to $\mathcal{P}(n)$)
 - A limit (with arrow pointing to $\mathcal{P}(1)$)
 - Partial compositions (with arrow pointing to the composition diagram)


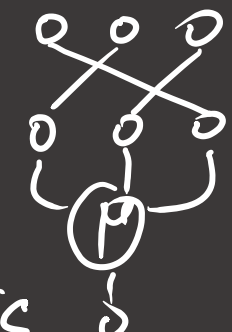
Operads and algebras

We just need Biproducts and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)

An operad is:

- $\{\mathcal{P}(n)\}$  Vector spaces
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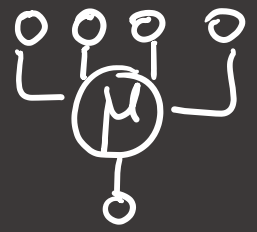


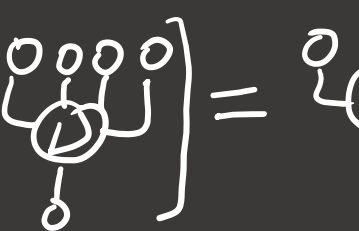
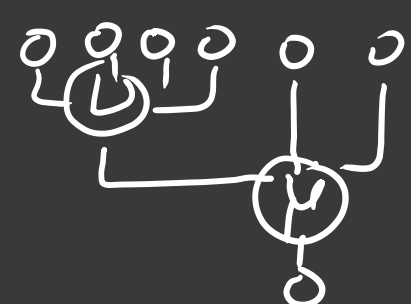
•  $\cdot \sigma =$ 
 Symmetric actions

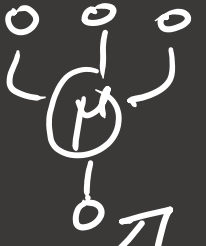
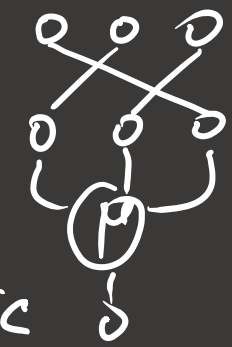
Operads and algebras

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 Symmetric actions

This gives a monad!

Operads and algebras

We just need Biproducts
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Take the symmetric monoidal category of vector spaces (this can be generalized)

An algebra of an operad is:

- a vector space A

Operads and algebras

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An algebra of an operad is:

- a vector space A

- $\vartheta_n : \mathcal{O}(n) \otimes A^{\otimes n} \rightarrow A$

$$\vartheta_n(\mu; a_1, \dots, a_n) = \mu(a_1, \dots, a_n)$$


Operads and algebras

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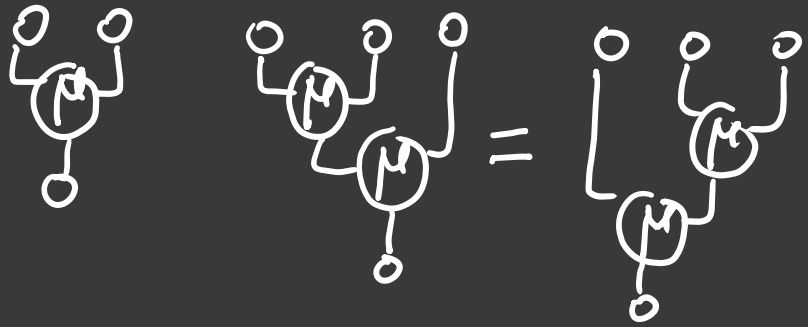
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 An algebra
is a representation
of the operad!

Operads and algebras: examples

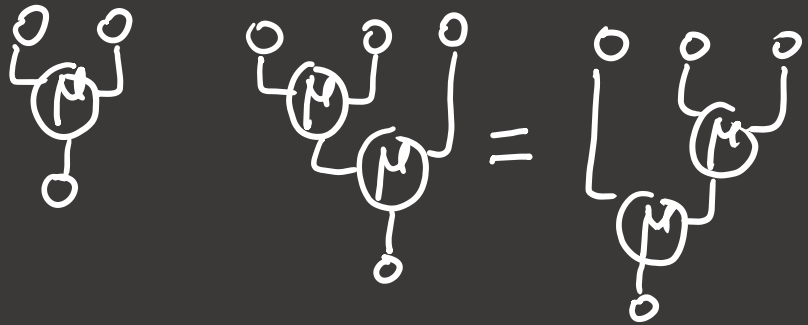
Ass



Associative algebras

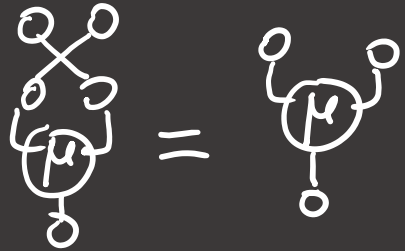
Operads and algebras: examples

Ass



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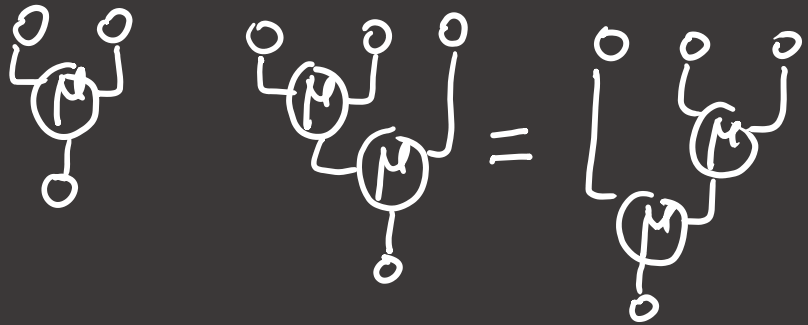
Com



Commutative algebras

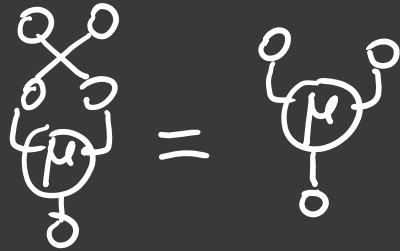
Operads and algebras: examples

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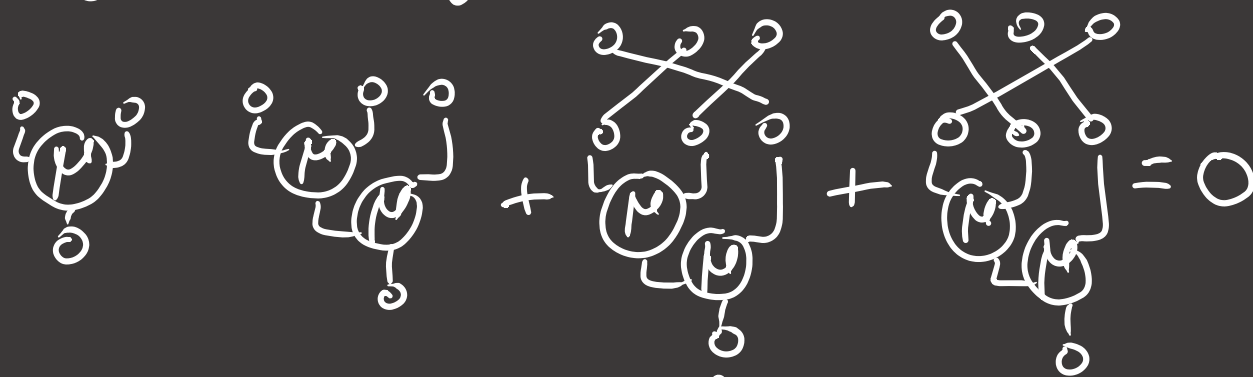
Associative algebras

Com



Commutative algebras

Lie



Lie algebras

Operads are **tangent** monads

“Operads are machines that produce tangent categories.”



Operads are tangent monads

This is the
new stuff

"Operads are
machines that produce
geometrical theories."



Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023



Theorem.

The monad associated to an operad is a coCartesian differential monad

Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023



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$\mathcal{P} \leftarrow \text{Operad}$

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Ikonicoff, Lanfranchi, Lemay 2023

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$$S_{\mathcal{P}} X = \bigoplus_m \mathcal{P}(m) \otimes_{S_m} X^{\otimes m}$$

Monad associated to \mathcal{P}

Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023

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Monad associated to \mathcal{P}

Differential
calculator

$$\partial: S_{\mathcal{P}} X \rightarrow S_{\mathcal{P}} (X \oplus X)$$

$$\partial(\mu; x_1, \dots, x_n) = \sum_{k=1}^n (\mu; (x_1, 0), \dots, (0, x_k), \dots, (x_n, 0))$$

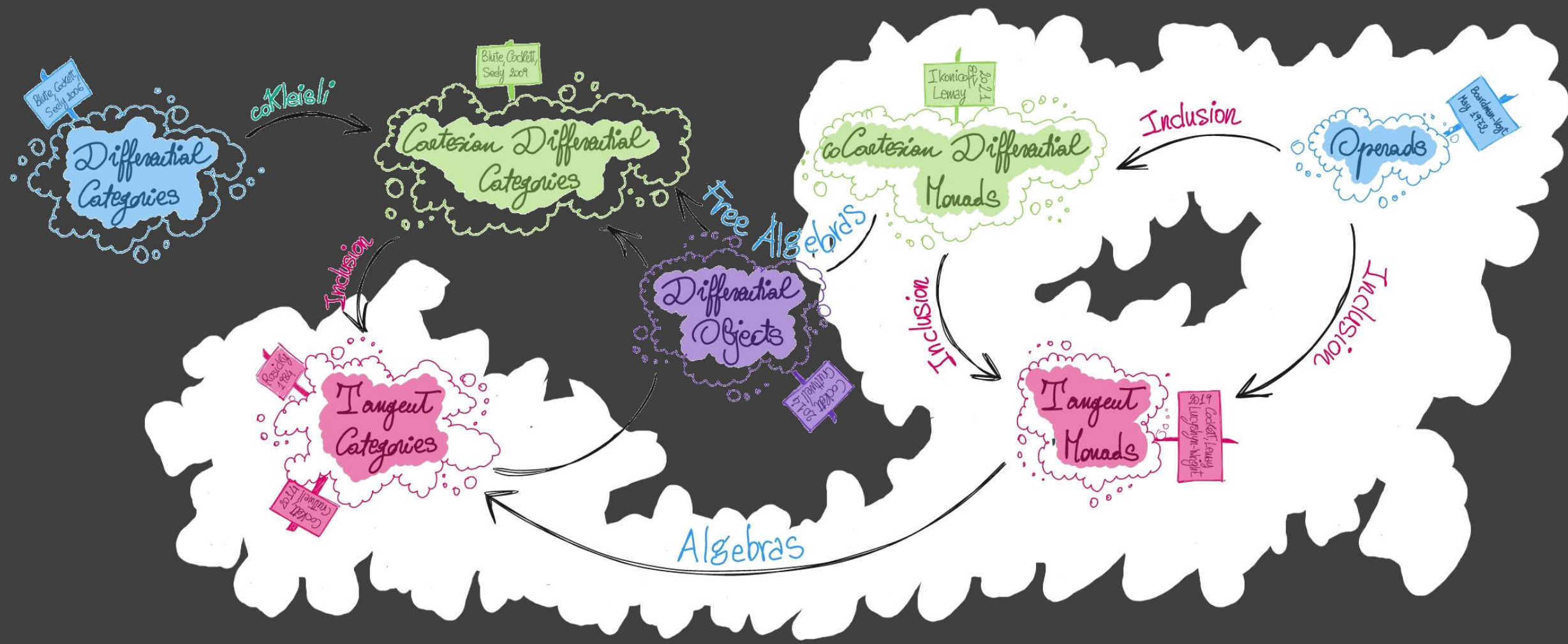
Operads are tangent monads



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Operads are tangent monads



Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.

Let P be an operad.

The categories Alg_P and Alg_P^{op} are tangent categories.

The main result of this talk!

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.

Let P be an operad. The tangent bundle functor $T: Alg_P \rightarrow Alg_P$ is the functor so defined:

$$TA := A \rtimes A$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

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$A \oplus A$ as a vector space

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Ikonicoff, Lanfranchi, Lemay 2023

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$A \oplus A$ as a vector space

$$\begin{aligned} \mu((a_1, b_1), \dots, (a_m, b_m)) &= \\ &= \sum_{k=1}^m (\mu(a_1, \dots, a_m), \mu(b_1, \dots, b_m)) \end{aligned}$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.

Let P be an operad. The tangent bundle functor $T^\circ: \mathit{Alg}_P^{op} \rightarrow \mathit{Alg}_P^{op}$ is the functor so defined:

$$T^\circ A := \mathit{Free}_A \Omega_A$$

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Ikonicoff, Lanfranchi, Lemay 2023

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A -module of
Kähler differentials
of A


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Free A -algebra
of an A -module

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Ikonicoff, Lanfranchi, Lemay 2023

Theorem.

Let P be an operad. The tangent bundle functor $T^\circ: \mathit{Alg}_P^{op} \rightarrow \mathit{Alg}_P^{op}$ is the functor so defined:

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Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

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$$\sum_{k=1}^m \mu(a_1, \dots, da_k, \dots, a_m)$$

New models of tangent categories

Cruttwell, Lemay 2023

Commutative & Unital Algebras

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Commutative & Unital Algebras

Cruttwell and Lemay showed that $(cAlg^{op}, T^o)$ describes some geometrical aspects of the algebraic geometry of affine schemes.

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New models of tangent categories

Ginzburg 2005

Associative & Unital Algebras

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Ginzburg described non-commutative algebraic geometry.

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tensor algebra

$$T^{\circ}A = \overbrace{\text{Tensor}}^{\text{mc}} A \underbrace{\Omega_A}_{\text{non-commutative}}$$

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This is the 1st model of non-commutative geometry described with tangent categories



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What next?

What about Lie algebras?
or Poisson algebras...

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Differential Bundles?
Connections? Cohomology?
Differential equations?

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Differential
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Differential Bundles?
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∞ -Operads \rightarrow ∞ -Geometry?
Functoriality?

Thanks.

The Rosický Tangent Categories of Algebras over an Operad
<https://arxiv.org/abs/2303.05434>

